TITLE SUBNORMALITY AND SOLUBLE FACTORISED GROUPS

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Summary

Throughout this summary the group $G = AXB$ is always a product of three abelian subgroups A, X and B.

In Chapter 1 we study finite 2-groups G, where A and B are elementary and X has order 2. We also assume that X normalises both A and B, and thus AX and XB are nilpotent of class at most 2. We show that when the order of G divides 2^{13} then G has derived length at most 3 ((1.4.2) and (1.6.1)). This supports the conjecture [see Introduction] on the derived length of a group which is expressible as the product of two nilpotent subgroups.

In Chapter 2 we consider some special cases of G where A, X and B are finite p-groups and X is cyclic. We obtain a bound for the derived length of G which is independent of the prime p and the order of X.

In Chapter 3 we find a bound for the derived length of a finite group G in terms of the highest power of a prime dividing the order of X when $A^X = A$, $B^X = B$ and X is subnormal in both AX and XB. The most general result is Theorem (3.5.1). If G is a finite p-group and X has order p we show that G has derived length at most 4 (Theorem (3.3.1)).

Further in Chapter 3 if $A^X = A$, $B^X = B$, $X \lhd^m AX$ and $X \lhd^m XB$ then a bound for the subnormal defect of X in G is given. When X has order p this bound depends only upon m (see (3.3.4)), and when X has order p^n and m is fixed then the subnormal defect of X in G can be bounded in terms of n (see the remark following Proposition (3.4.2)).

Chapter 4 shows how some results from Chapters 2 and 3 can be generalised to infinite groups. Theorem (4.3.1) shows that when A and B are p-groups of finite exponent, X has order p^n, $A^X = A$, $B^X = B$, $X \lhd^2 AX$ and $X \lhd^2 XB$ then G is a locally finite group. Proposition (4.2.2) and Corollary (4.2.3) then enable some of the results about finite groups to be applied.

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Introduction

Motivation.

Any group G which is expressible as the product AB of two abelian subgroups A and B has been shown to be metabelian by Itô [I2]. Following results by P.Hall and G.Higman [Ha&Hi] it has been conjectured that a finite group $G = HK$ with subgroups H and K which are nilpotent of classes c and d respectively will have derived length at most $c + d$. In this thesis we find a bound for the derived length of certain groups which are the product of an abelian group with a nilpotent group. Some results obtained are in support of the conjecture mentioned above.

Recall that the *(subnormal) defect* of a subnormal subgroup X in a group G is the shortest length d of a series of subgroups

$$X = X_0 \lhd X_1 \lhd \dots \lhd X_d = G.$$

Then we write $X \lhd^m G$ for any $m \geq d$.

H.Wielandt [Wi3] has shown that

> *Suppose that a finite group* $G = HK$ *where* H *and* K *are subgroups and let* X *be a subgroup of* $H \cap K$ *with* X *subnormal in* H *and* K. *Then* X *is subnormal in* G.

This result was proved by reduction to the case when G is a p-group, and no function that bounds the subnormal defect of X in G in terms of its defects in H and K is known in general.

In order to ascertain whether it is likely that Wielandt's result holds in infinite groups we bound the defect of X in G for some finite groups G.

S.E.Stonehewer has shown [St1] that there is a connection between special cases of these two problems. In particular he proves

> *Let* $G = HK$ *be a group with subgroups* H *and* K. *Suppose* X *is a subgroup of* G *with* $X \lhd^{m} H$ *and* $X \lhd^{m} K$. *Let* A *be a normal abelian subgroup of* G *and suppose that* $AX \lhd^{m_1} G$ *and* $X(H \cap A)(K \cap A) \lhd^{m_2} G$, *then* $X \lhd^{2m+m_1+m_2} G$.

If G is a soluble group and H and K are nilpotent we can take A to be the smallest non-trivial term in the derived series of G. We shall see that in certain cases when we have obtained a bound for the derived length of our group G (see Theorems (3.3.1) and (3.4.1)) inductive arguments may be used to find a bound for the subnormal defect of X in G.

We tackle special cases of the afore-mentioned problems side by side. Our interest here lies in the study of groups of the form $G = HK$, where H and K are nilpotent subgroups of G and $H = AX$, $K = XB$ for subgroups A, X, B of G. Since it is not known whether the conjecture concerning the derived length of a finite group $G = HK$, with nilpotent subgroups H and K, holds in the case when H has class 1 and K has class 2 we assume A, X, B to be abelian and tackle this case.

Solubility of products of groups

In 1951 Itô [I1] proved that a finite group G = HK which is the product of subgroups H and K, where H is nilpotent and K is either abelian or a p-group, is soluble (see also Gruenberg [Gru]). Wielandt [Wi1] then showed that a finite group $G = G_1G_2...G_n$ with pairwise permutable nilpotent subgroups $G_1, G_2, ..., G_n$ is soluble if G_iG_j is soluble for all i and j. He also used Itô's result to prove that a finite product of pairwise permutable subgroups, one of which is nilpotent and the others being either abelian or p-groups, is soluble.

Many results concerning the solubility of a finite factorised group followed. Some of these results give a bound for the derived length of the group, others prove the solubility of certain products of groups.

In 1953 Huppert [H1] proved that a finite group G = HK is soluble if H and K are subgroups of G, where H is dihedral and K is i) abelian, or ii) a p-group, or iii) dihedral. Huppert and Itô [HI] then showed that G is soluble when K is nilpotent and H contains a cyclic subgroup of index 2. They also proved the solubility of G when K is cyclic of odd order, H' is nilpotent and H contains a normal Hall 2'-subgroup. Huppert [H2] went on to show that G is soluble when H is abelian and K has a cyclic normal subgroup of index 2.

In 1955 Itô [I2] proved

> *Let* G = AB *be a group (finite or infinite) with abelian subgroups* A *and* B, *then* G *is metabelian.*

We refer to this result as (*) throughout this introduction.

Scott [Sc1] then showed that a finite group G = HK with subgroups H and K is soluble when one of the following is true :-

i) H is nilpotent and K is either abelian or Hamiltonian.

ii) H is nilpotent, |H| is odd and there is a subgroup L of K such that

$|K:L| = 2$ and every subgroup of L is normal in K.

iii) H is cyclic and there is a normal subgroup L of K such that $|K:L| = 2$ or 3 and every subgroup of L is normal in K.

iv) H and K are both either dicyclic or dihedral.

In 1958 Wielandt [Wi2] was able to prove that a finite group which is expressible as the product of two nilpotent subgroups with coprime orders is soluble. Then Kegel [Ke1] showed that the assumption that these two subgroups have coprime order is unnecessary. Scott [Sc1] proved some special cases of Kegel's results and showed that a finite group G = HK is soluble of derived length at most 4 when H is abelian and K is Hamiltonian (ie. K is non-abelian and every subgroup of K is normal in K). He also proved that G is still soluble if one (but not both) of H and K is infinite. In this case a slightly larger bound for the derived length of G is obtained.

Let $\mathfrak{X}$ be a class of groups, then a group G is a locally $\mathfrak{X}$-group if every finitely generated subgroup of G is contained in an $\mathfrak{X}$-group. In 1965 Kegel [Ke2] extended earlier work (see Kegel [Ke1] and Wielandt [Wi1]) to show that a locally finite group $G = G_1G_2...G_n$ which satisfies the minimal condition for subgroups, where $G_1, G_2,..., G_n$ are locally nilpotent groups such that $G_iG_j = G_jG_i$ for all i and j, is soluble. Berkovic [B] produced a further generalisation of Wielandt's result [Wi2] in 1966. He proved that a group G which is the product of pairwise permutable subgroups $G_0, G_1,..., G_n$ with pairwise coprime orders is soluble when G_0 is 2-decomposable and $G_1,..., G_n$ are nilpotent of odd order. Doktorov [D] also weakens Wielandt's hypotheses and proves results on p-solubility, where a group is said to be p-soluble if it possesses an abelian series such that every factor group of the series is either a p-group or a p'-group.

Let C be the class of finite groups L such that all 2-subgroups of L are normal in L. A result by Scott and Gross [SG] shows that if G = HK, where H and K belong to C, then G is soluble.

There has also been much interest in finite groups that are the product of two subgroups, at least one of which contains a nilpotent subgroup of index 2. In 1976 Knopp [Kn] showed that G = HK is soluble when H contains a nilpotent subgroup of index 2 and K is cyclic. He also proved that G is soluble if H is Dedekind (i.e. all subgroups of H are normal in H) and K contains a Dedekind subgroup of index 2.

Carr [Ca] showed that G = HK is soluble when H is abelian and K contains a nilpotent subgroup of index 2. This result was extended by Walls [Wa] to the case when H is Dedekind.

Finkel [F1] also showed that a finite group G = HK is soluble if there exist subgroups H_0, K_0 having index 2 in H and K respectively where H_0 is nilpotent and K_0 is a p-group. He then proved [F2] that G is soluble if H has a nilpotent subgroup of index 2 and K is Dedekind. In two papers, [FL1] and [FL2], of Finkel and Lundgren a finite group G was also shown to be soluble if H contains a nilpotent subgroup, H_0, of index 2, K is nilpotent and one of the following holds:-

1) $|H_0|$ is odd.
2) If $H \leq M < G$ then $|M : H|$ is odd.
3) H is maximal in G.
4) The Sylow 2-subgroup of K is abelian.
5) Every 2-subgroup of K is normal in K.
6) $|K|$ is odd.

Kazarin [Ka] showed that when H and K both contain nilpotent subgroups of index at most 2 then G is soluble. The conditions on the indices of the nilpotent subgroups are necessary as shown by an example of Kazarin. In the same paper Kazarin proves that G is soluble when H is nilpotent and K contains a normal nilpotent subgroup K_0 of index at most 4 and $\exp(K/K_0) \leq 2$ as long as H and K have coprime orders.

It has been recently proved by Chernikov (see [Ch1] and [Ch2]) that a group factorised by two subgroups, that are finite over their centres, is almost soluble, i.e. contains a soluble subgroup of finite index.

Derived length of soluble factorised groups.

Another consequence of Itô's result (*) is that it gives a bound on the derived length of the product $G = HK$, given some information about H and K. Following this Hall and Higman [Ha&Hi] proved that any finite soluble group G which can be expressed as the product of nilpotent subgroups $H_1, H_2, \ldots, H_n$, where $(|H_i|, |H_j|) = 1$ for $i \neq j$, has derived length at most $c_1 + c_2 + \ldots + c_n$, where H_i has class c_i for all i. These results led to the conjecture that *any* finite group which can be expressed as the product of a nilpotent group of class c with a nilpotent group of class d will be soluble of derived length at most $c + d$. However, no bound for the derived length of G in terms of c and d is known in general.

Gross [Gro] showed that if $G = HK$ is a finite group which is the product of nilpotent subgroups H and K, where H and K have classes c and d respectively, then the derived length of $G/\Phi(G)$ is at most $c + d$, where $\Phi(G)$ is the Frattini subgroup of G. Around the same time, Pennington [P] also showed that $G^{(c+d)}$ is a nilpotent π-group where π is the set of primes dividing $|H|$ and $|K|$. In the same paper she also shows that the Fitting subgroup F(G) of G inherits a factorisation from G i.e.

$$F(G) = (H \cap F(G))(K \cap F(G)).$$

It follows that in order to obtain a bound for the derived length of G as a function of c and d it is sufficient to find a bound for the derived length of a factorised p-group.

The problem of finding the derived length of a finite group $G = AX$, with abelian subgroup A and nilpotent subgroup X, in terms of the class of X and the order of X′ was posed in Question 4.58 of [Ko]. Scott had earlier obtained a bound in the case when X is nilpotent of class 2 [Sc3]. In 1983 Zaitsev [Z] proved that the infinite group G has derived length at most $2 + 3\alpha$ when X′ is finite of order $p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_n^{\alpha_n}$ where the p_i's are distinct primes, $\alpha_i \geq 1$ for all i and $\alpha = \alpha_1 + \alpha_2 + \ldots + \alpha_n$.

Subnormality in factorised groups

In 1977 Maier [M] proved that in a finite group $G = HK$, where H and K are subgroups and X is a soluble subgroup of $H \cap K$, if X is subnormal in both H and K then X is subnormal in G. Then Wielandt [Wi3] showed that the assumption that X be soluble is unnecessary. The result of Wielandt was proved by reduction to the case when G is a finite p-group, and hence no formula bounding the subnormal defect of X in G in terms of its defects in H and K was obtained.

Wielandt's result has been shown to hold for certain classes of infinite groups by Stonehewer in [St1]. In particular Wielandt's result is true for the class of nilpotent-by-abelian groups, and also for all periodic groups which are nilpotent-by-abelian-by-finite. Suppose that $G = HK$ is nilpotent-by-abelian-by-finite and there is a normal subgroup B of G such that B is an extension of a nilpotent group of class c by an abelian group, and $|G/B|$ is finite of order less than or equal to n. In this situation Stonehewer has shown that an integer $f = f(m, c, n)$ exists such that $X \lhd^f G$, where m is the subnormal defect of X in H and of X in K.

Stonehewer has also proved (see [St2]) that if $G = HK$ is a group with subgroups H, K and X, where X is subnormal in both H and K, then X is subnormal in G provided that one of the following holds :-

i) G is nilpotent-by-polycyclic-by-finite.

ii) G satisfies the minimal condition modulo a nilpotent normal subgroup.

iii) G is a soluble minimax group.

Statement of main results

In this thesis we prove several results related to the previous sections. In this section we give a resume of the main results and methods.

Chapter 1 deals mainly with finite 2-groups $G = AXB$ where A, X and B are elementary abelian and X has order 2. We also assume that X normalises A and B. In Theorem (1.4.2) and Proposition (1.6.1) we show that if $A \cap B = 1$ and $|G|$ divides 2^{13} then G has derived length at most 3. This is one case when the conjecture concerning the derived length of a product of two nilpotent groups (mentioned in the section *Derived length of soluble factorised groups*, in this introduction) holds. In the proof of these results we use the following fact

> *If a cyclic group,* X, *of order* p^n *acts on a finite elementary abelian p-group,* A, *then* A *can be written as a direct product of indecomposable* X*-submodules, each having rank at most* p^n.

where the rank of an elementary abelian p-group is defined to be its dimension when viewed as a vector space over the field $\mathbb{F}_p$ of p elements.

This can easily be seen to be true by viewing A as a finite dimensional vector space over $\mathbb{F}_p$ and considering the Jordan Normal Form of the matrix for x as a linear transformation on A, where $\langle x \rangle = X$.

I am grateful for help from Professor G. Busetto, A. Leeves and Dr. S.E. Stonehewer in proving Lemmas (1.5.1) and (1.5.2) and Proposition (1.6.1).

In Chapter 2 we study finite groups $G = AXB$ where A, X and B are abelian p-groups and X is cyclic. When A and B are elementary, $A^X = A$, $B^X = B$, $X \lhd^2 AX$ and $X \lhd^2 XB$ we tackle the cases $p = 2$ and $p \neq 2$ separately. In each case a bound for the derived length of G is given (Lemma (2.2.1)ii) and Lemma (2.2.2)ii)). We also reduce the problem of bounding the subnormal defect of X in G to the case when $|X| = p$ (Lemma (2.2.1)i) and Lemma (2.2.2)i)).

Then we turn our attention to finite p-groups $G = AXB$ with abelian subgroups A, X and B where X and B are cyclic. We begin by considering the case when $|X| = p$ and XB is a nilpotent subgroup of class 2 (Proposition (2.3.1)). In this case we show that the derived length of G is at most 3, which supports the conjecture on derived length mentioned earlier. When $|X| = p^n$ we also assume that X normalises A and B and show that the derived length of G is at most 4 when XB is a nilpotent subgroup of class 2 (Proposition (2.3.3)). A bound for the subnormal defect of X in G is also given when $X \lhd^2 AX$.

In Chapter 3 we consider the problems of subnormal defect and derived length side by side. In particular we study finite groups $G = AXB$ with abelian subgroups A, X and B. If X normalises A and B and has subnormal defect at most m in both AX and XB we find a bound for the derived length of G in terms of the highest power of a prime dividing $|X|$. When X is a p-group we use an inductive argument to obtain a bound for the subnormal defect of X in G as a function of m, p, n and the derived length of G, where X has order p^n (Proposition (3.4.2)). If $n = 1$ this bound depends only upon m (see Remark following Theorem (3.3.4)). If m is fixed then the subnormal defect of X in G can be bounded by a function of n, for all primes p (see the Remark following (3.4.2)).

In Chapter 4 we consider infinite groups and show how some of the results of Chapters 2 and 3 can be extended to infinite groups. Theorem (4.3.1) shows that when $G = AXB$ is a group with abelian p-subgroups A, X and B of finite exponent, where X has order p^n, $A^X = A$, $B^X = B$, $X \lhd^2 AX$ and $X \lhd^2 XB$, then G is a locally finite p-group. Proposition (4.2.2) and Corollary (4.2.3) then enable some results from Chapters 2 and 3 to be applied.

Notation and Definitions

Throughout this thesis capital letters denote groups and lower case letters denote elements of a group or a set. We always use p to denote a prime number.

Let g, h be elements of a group G. Define the *conjugate* of h by g to be $h^g = g^{-1}hg$. Let H and K be subgroups of a group G, then H^K is defined to be the subgroup

$$H^K = \langle h^k \mid \text{for all } h \in H \text{ and } k \in K\rangle.$$

When K = G, then H^G is the *normal closure* of H in G.

The *commutator* [g,h] is defined to be $g^{-1}h^{-1}gh$, and the commutator

$$[g_1, g_2, g_3, \ldots, g_n] = [[\ldots[[g_1, g_2], g_3], \ldots], g_n]$$

is defined inductively, where g, h, $g_1, g_2, g_3, \ldots, g_n$ are group elements. We sometimes use the notation $[g, {}_nh] = [g, h, h, \ldots, h]$, where h appears n times, for brevity.

Let H and K be subgroups of a group G. The subgroup generated by all commutators [h, k], where $h \in H$ and $k \in K$ is written [H, K]. The group $[H_1, H_2, H_3, \ldots, H_n] = [[\ldots[[H_1, H_2], H_3], \ldots], H_n]$ is then defined inductively. The notation $[H, {}_nK]$ is sometimes used to denote [H, K, K,..., K], where K appears n times.

The *derived subgroup* G′ of G is defined by G′ = [G, G]. The nth term $G^{(n)}$ in the derived series of G is then defined inductively by $G^{(n)} = [G^{(n-1)}, G^{(n-1)}]$, where $G^{(1)} = G'$. Sometimes the notation $G'' = G^{(2)}$, and $G''' = G^{(3)}$ is also used. G is said to be *soluble of derived length d* if $G^{(d)} = 1$ and $G^{(d-1)} \neq 1$.

A group G is *nilpotent of class c* if c is the smallest integer such that $[G, {}_cG] = 1$.

Recall that an *elementary abelian p-group* is a group G isomorphic to a direct product of cyclic groups of order p.

Let π be a set of primes, then π' denotes the set of primes that do not belong to π. If G is a finite group with a subgroup H such that $|H|$ is a π-number and $|G:H|$ is a π'-number then H is called a *Hall π-subgroup* of G. When $\pi = \{p\}$ H is called a *Sylow p-subgroup* of G. The set of Sylow p-subgroups of G is denoted by $Syl_p G$. The set of Hall π-subgroups of G is denoted by $Hall_\pi(G)$.

The following results are used many times throughout this thesis and are quoted here for easy reference.

Itô's Result

Let G = AB be a group (finite or infinite) with abelian subgroups A and B, then G is metabelian.

Proof See Scott [Sc3] (13.3.2).

Dedekind's Intersection Lemma

Let H, K, L be subgroups of a group and suppose that $K \subseteq L$. Then $(HK) \cap L = (H \cap L) K$.

Proof See Robinson [R3] (1.3.14).

The Three Subgroup Lemma (Kaluzin, P.Hall)

Let H, K, L be subgroups of a group G. If two of the commutator subgroups [H, K, L], [K, L, H], [L, H, K] are contained in a normal subgroup of G, then so is the third.

Proof See Robinson [R3] (5.1.10).

List of notation

$H \leq G$	H is a subgroup of the group G
$H < G$	H is a proper subgroup of the group G
$H \lhd G$	H is a normal subgroup of the group G
G/N	the group whose elements are gN, where G is a group, $N \lhd G$ and $g \in G$
$H \lhd^m G$	H is a subnormal subgroup of the group G, having subnormal defect at most m in G
$H \subseteq S$	H is a subset of the set S
$\langle S \mid S \subseteq G \rangle$	the smallest subgroup of G containing S
$s \in S$	s is an element contained in the set S
$X \setminus Y$	$\{x \in X \mid x \notin Y\}$
$\lvert H \rvert$	the cardinality of the set H, or the order of the group H
$\lvert g \rvert$	the order of the group element g
HK	the set of elements of the form hk, where $h \in H$ and $k \in K$
d(G)	the derived length of the group G
$\lvert G:H \rvert$	the number of cosets of the subgroup H in the group G
Z(G)	the centre of the group G (ie. the set of all elements in G that commute with every element of G)
$C_G(g)$	the *centraliser* of g in G (the subgroup of the group G consisting of all elements $h \in G$ such that $gh = hg$)
$C_G(H)$	the *centraliser* of H in G (the subgroup of the group G consisting of all elements $g \in G$ such that $gh = hg$ for all $h \in H$)
$N_G(H)$	the *normaliser* of H in G (the largest subgroup of the group G in which the subgroup H is normal)
$\Phi(G)$	the Frattini subgroup of G (the intersection of all maximal subgroups of the group G)

$F(G)$	the Fitting subgroup of G (the group generated by all normal nilpotent subgroups of the group G)
$\Omega_1(G)$	the subgroup of the group G generated by all elements in G of order p
$\mathbb{Z}$	the set of integers
$\mathbb{N}$	the set of positive integers
π	a set of primes
π'	all primes that do not lie in the set π (when $\pi = \{p\}$ write p' for $\{p\}'$)

Chapter 1:

Finite products of three elementary abelian 2-groups

(1.1) Introduction.

In this chapter we study finite p-groups G of the form

$$G = AXB$$

where A, X and B are elementary abelian p-subgroups of G, X has order p and X normalises both A and B. We show that there are some cases when G has derived length less than or equal to 3 if AX is nilpotent of class at most 2.

Some preliminary lemmas are proved for a general prime p and these are used to show that G has derived length at most 3 when [A, X] has order p, $X \lhd^2 AX$ and $X \lhd^2 XB$. Then we turn our attention to the case where the prime $p = 2$ and show that the derived length of G is less than or equal to 3 when [A, X] has order 4.

Theorem (1.4.2) is a corollary of these 2 propositions and shows that the derived length of G is at most 3 when $A \cap B = 1$ and the order of G divides 2^{12}. This result was originally obtained by a series of long calculations which suggested the more general lemmas above.

I am grateful for help from Professor G. Busetto, A Leeves and Dr. S. E. Stonehewer in proving Lemmas (1.5.1) and (1.5.2) and Proposition (1.6.1). The two lemmas are general results concerning finite factorised groups G = AXB where A, X and B are abelian and X normalises A and B. These lemmas are used in Proposition (1.6.1) to show that Theorem (1.4.2) can be extended to cover the case when G has order 2^{13}.

Recently it has been shown by A. Leeves (Warwick University PhD. Thesis -to appear) that all finite 2-groups of the form G = AXB where A, X and B are elementary abelian subgroups with $|X| = 2$ and $A^X = A$ have derived length at most 3, using methods suggested by the case when G has order 2^{13}. He has also been able to show that a finite p-group $G = AXB$ has derived length at most 3 when $p \neq 2$ is prime, A, X, B are abelian subgroups of G, X is cyclic, $A^X = A$, $B^X = B$, $X \triangleleft^2 AX$ and $X \triangleleft^2 XB$.

(1.2) Technical Results.

The first 3 technical results that we prove assume that A and B are abelian p-subgroups of G.

Lemma (1.2.1)

Let $G = AXB$ be a finite group with abelian p-subgroups A, X and B. Suppose also that X has order p and $A^X = A$. Then $\langle A_1, B\rangle \subseteq AB$ where $A_1 = [A, X]$.

Proof

We argue by induction on $|G|$.

Note that G is a p-group since it is the product of AX and B, both of which are finite p-groups. It follows that G has non-trivial centre.

Let $\langle x\rangle = X$.

Suppose that the result is false and let G be a counterexample of minimal order.

Case i). $Z(G) \cap AB = 1$.

Let N be a minimal normal subgroup of G. Then $N \leq Z(G)$ and we may assume without loss of generality that $N = \langle axb\rangle$ for some $a \in A$ and $b \in B$.

$$1 = [axb, B] = [ax, B]$$

implies that

$$\langle ax\rangle^G \leq AX.$$

Hence we can find a minimal normal subgroup N_1 of G with $N_1 \leq AX$. Then $N_1 \leq Z(G)$ and $Z(G) \cap AB = 1$ imply together that

$$N_1 = \langle a'x^\alpha\rangle$$

for some $a' \in A$ and $\alpha \not\equiv 0 \bmod p$.

It follows that

$$1 = [a'x^\alpha, A] = [x^\alpha, A]$$

and thus

$$A_1 = [X, A] = 1$$

since $X = \langle x^\alpha\rangle$.

Hence $\langle A_1, B\rangle = B \subseteq AB$. This is a contradiction.

Case ii). $Z(G) \cap AB \neq 1$.

Then $1 \neq ab \in Z(G)$ for some $a \in A$ and $b \in B$. Note that

$$1 = [ab, b] = [a, b]^b$$

implies that $[a, b] = 1$ and so $(ab)^i = a^ib^i$ for all $i \in \mathbb{Z}$.

It follows that $\langle ab\rangle$ is a normal subgroup of G contained in AB. Thus we can find a minimal normal subgroup, N, of G with $N \subseteq AB$.

Since $|G/N| < |G|$ we see that

$$\frac{\langle A_1, B\rangle N}{N} \subseteq \frac{AN}{N}\frac{BN}{N}$$

Thus $\langle A_1, B\rangle N \subseteq ANBN = ANB = AB$ and a contradiction is obtained in this case also.

Hence G cannot be a counterexample and so $\langle A_1, B\rangle \subseteq AB$ must be true.

$\square$

An immediate corollary of this result is

Corollary (1.2.2)

Let $G = AXB$ be a finite group with abelian p-subgroups A, X and B, where X has order p. Suppose also that $A^X = A$ and $X \lhd^2 AX$. Then $[A_1, B]$ is an abelian normal subgroup of G, where $A_1 = [A, X]$.

Proof

It is clear that AX is nilpotent of class at most 2 since $X \lhd^2 AX$. Thus

$$A_1 \leq Z(AX).$$

Then $[A_1, G] = [A_1, AXB] = [A_1, B]$ and $[A_1, B]$ is normal in G (since $[A_1, G] \lhd \langle A_1, G\rangle = G$).

$$\langle A_1, B\rangle = \langle A_1, B\rangle \cap AB \quad \text{by (1.2.1)}$$
$$= (\langle A_1, B\rangle \cap A)\, B \quad \text{by Dedekind's}$$

intersection lemma (see Notation).

Hence $\langle A_1, B\rangle$ is expressible as a product of two abelian groups and so is metabelian by Itô (see Notation). Thus

$$[A_1, B] \leq \langle A_1, B\rangle'$$

is abelian.

$\square$

We now assume that X normalises A and B and that X has subnormal defect 2 in both AX and XB, and use the symmetry of A and B in G to prove

Lemma (1.2.3)

Let $G = AXB$ be a finite group with abelian p-subgroups A, X and B, where X has order p and $A \cap B = 1$. Suppose also that $A^X = A$, $B^X = B$, $X \lhd^2 AX$ and $X \lhd^2 XB$. Then

$$N = [A_1, B][A, B_1]$$

is an abelian, normal subgroup of G, where $A_1 = [A, X]$ and $B_1 = [B, X]$.

Proof

The normality of N in G follows from (1.2.2).

Note that $X \lhd^2 AX$ and $X \lhd^2 XB$ imply that AX and XB are both nilpotent of class at most 2, since $A^X = A$ and $B^X = B$. Thus $A_1 \leq Z(AX)$ and $B_1 \leq Z(XB)$.

We use Ito's method ([Sc3] (13.3.2)) to show that N is abelian.

Let $a_1 \in A_1$, $a_2 \in A$, $b_1 \in B_1$ and $b_2 \in B$ and let

$$b_2^{a_2} = a_3x^{\varepsilon}b_3, \quad a_1^{b_1} = b_4a_4$$

for some $a_3, a_4 \in A$ and $b_3, b_4 \in B$ where $X = \langle x \rangle$ and $\varepsilon \in \{0, 1, \ldots, p-1\}$. (Note that $a_1^{b_1} \in AB \cap BA$ by Lemma (1.2.1) and symmetry of A and B).

Then

$$\begin{aligned}
[a_1, b_2]^{a_2b_1} &= [a_1, b_2^{a_2}]^{b_1} \\
&= [a_1, a_3x^{\varepsilon}b_3]^{b_1} \\
&= [a_1, b_3]^{b_1} \quad \text{since } A_1 \in Z(AX) \\
&= [a_1^{b_1}, b_3] \\
&= [b_4a_4, b_3] \\
&= [a_4, b_3].
\end{aligned}$$

Also

$$\begin{aligned}[a_1, b_2]^{b_1 a_2} &= [a_1{}^{b_1}, b_2]^{a_2} \\ &= [b_4a_4, b_2]^{a_2} \\ &= [a_4, b_2]^{a_2} \\ &= [a_4, b_2{}^{a_2}] \\ &= [a_4, a_3x^{\epsilon}b_3] \\ &= [a_4, b_3][a_4, a_3x^{\epsilon}]^{b_3}. \qquad (*)\end{aligned}$$

Now $a_1{}^{b_1} \in C_G(x)$ implies that $b_4a_4 \in C_G(x)$.

Claim : $C_G(x) = C_B(x)XC_A(x)$

Let $bx'a \in C_G(x)$ where $b \in B$, $a \in A$ and $x' \in X$. Then

$$bx'a = b^x x' a^x$$

and so

$$[x, b]x' = x'[x, a^{-1}].$$

However $A_1 \leq Z(AX)$ and $B_1 \leq Z(BX)$ and thus we see that

$$[x, b] = [x, a^{-1}] \in A \cap B = 1.$$

Hence $b \in C_B(x)$ and $a^{-1} \in C_A(x)$. It follows that

$$C_G(x) \subseteq C_B(x)XC_A(x).$$

The reverse inclusion is clear and so the claim is true.

Then $b_4a_4 \in C_G(x)$ implies that $b_4, a_4 \in C_G(x)$ and so $(*)$ becomes

$$[a_1, b_2]^{b_1 a_2} = [a_4, b_3].$$

Thus

$$[a_1, b_2]^{b_1 a_2} = [a_1, b_2]^{a_2 b_1}$$

and this is enough to prove that N is abelian by Corollary (1.2.2).

$\square$

We use the following corollary of (1.2.3) extensively in the rest of this chapter.

Corollary (1.2.4)

Let $G = AXB$ be a finite group with elementary abelian p-subgroups A, X and B where X has order p and $A \cap B = 1$. Suppose also that $A^X = A$, $B^X = B$, $X \lhd^2 AX$ and $X \lhd^2 XB$. Then

$$N = [A_1, B][A, B_1]$$

is a normal elementary abelian p-subgroup of G, where $A_1 = [A, X]$ and $B_1 = [B, X]$.

Proof

By Lemma (1.2.3) it is sufficient to show that N has exponent dividing p (i.e. every element of N has order dividing p).

Since N is abelian, N will have exponent dividing p if and only if $[A_1, B]$ and $[A, B_1]$ both have exponent dividing p.

Recall that Lemma (1.2.1) implies

$$\langle A_1, B\rangle = (\langle A_1, B\rangle \cap A)B$$

(by Dedekind, (see Notation)) and so $\langle A_1, B\rangle$ is a product of 2 abelian groups both of which have exponent dividing p. It follows that $\langle A_1, B\rangle'$ has exponent dividing p since a result of Holt & Howlett [Ho&Ho] states that

> *Let $G = HK$ be a group with abelian subgroups H and K. Suppose that H and K both have exponent dividing p, where p is prime, then G' has exponent dividing p.*

Then $[A_1, B] \leq \langle A_1, B\rangle'$ implies that the exponent of $[A_1, B]$ must divide p.

Similarly $[A_1, B]$ has exponent dividing p and the result follows.

$\square$

This completes the proof of most of the technical results that we use in the remainder of this chapter.

(1.3) A result for a general prime, p.

We now go on to show that if $G = AXB$ is a p-group with subgroups AX and B such that AX is nilpotent of class at most 2 and B is abelian then there are some cases when G has derived length less than or equal to 3. In particular we assume that A and B are elementary abelian subgroups, X is a subgroup of order p normalising both A and B, $X \lhd^2 AX$ and $X \lhd^2 XB$, and some additional conditions.

The following result is proved for a general prime p.

Proposition (1.3.1)

Let $G = AXB$ be a finite group with elementary abelian p-subgroups A, X and B where X has order p, $A^X = A$ and $B^X = B$. Suppose also that $X \lhd^2 AX$, $X \lhd^2 XB$ and $|A_1| \le p$ where $A_1 = [A, X]$.

Then G has derived length at most 3.

Proof

We may assume that AX is non-abelian else $G'' = 1$ by Itô's result (see Notation).

Hence we may assume that $A \cap X = 1$ and $|A_1| = p$.

Let $B_1 = [B, X]$.Then $G/A_1{}^G$ and $G/B_1{}^G$ are both metabelian by Itô (see Notation) and thus

$$\frac{G}{A_1^G \cap B_1^G}$$

is also metabelian.

Hence it is sufficient to prove that $A_1{}^G \cap B_1{}^G$ is abelian. First we show that we may assume that $A \cap B = 1$.

Suppose that $A \cap B \neq 1$. Then one of the following cases occurs.

i) $A_1 \cap B \neq 1$
ii) $A_1 \cap B = 1$

In case i) we have that $A_1 \leq B$ since A_1 has order p. Thus $A_1{}^G = A_1$ which is abelian and so G has derived length at most 3 in this case.

In case ii) there is a complement A^* of $A \cap B$ in A containing A_1 (by the theory of vector spaces). Then A^* is normalised by X and we can replace A by A^*.

Thus we can assume that $A \cap B = 1$.

G is a product of 2 finite p-groups (namely AX and B) and so is itself a finite p-group. Thus G is nilpotent and satisfies the normaliser condition.

Hence we can construct a chain of distinct subgroups of G, each one being normalised by the next subgroup in the chain, as follows.

$$XB \lhd \langle a_1\rangle XB \lhd \langle a_1, a_2\rangle XB \lhd \langle a_1, a_2, a_3\rangle XB \lhd \ldots \lhd AXB$$

where $1 \neq a_i \in A$ for each i and $a_i \notin \langle a_1,\ldots,a_{i-1}\rangle$.

Let k be minimal such that a_k does not commute with X and let $A_2 = \langle a_1,\ldots,a_{k-1}\rangle$. Then $[A_2, X] = 1$ and

$$A_1 = \langle [a_k, x]\rangle \leq A_2XB = L \text{ say.}$$

By Itô (see Notation) L is metabelian since both A_2X and B are abelian subgroups of G. Hence $A_1, X, B \leq L$ implies that

$$[A_1, B]B_1 \leq L'$$

is abelian.

We notice that

$$\begin{aligned} A_1{}^G \cap B_1{}^G &= A_1{}^B \cap B_1{}^A \\ &= A_1[A_1, B] \cap B_1[B_1, A] \\ &\leq (A_1 \cap B_1[B_1, A][A_1, B])[A_1, B] \end{aligned}$$

by Dedekind (see Notation) and that this is abelian by (1.2.3), because $B_1[A_1, B]$ is abelian and so $(A_1 \cap B_1[B_1, A][A_1, B])$ is an abelian group all of whose elements commute with every element of the abelian group $[A_1, B]$.

Thus G has derived length at most 3 in this case also.

□

Note that if $G = AXB$ is a finite 2-group with elementary abelian subgroups A, X and B where X has order 2 and normalises A and B then we can deduce that $X \lhd^2 AX$ and $X \lhd^2 XB$. Thus in the results of Sections (1.2) and (1.3) these hypotheses are unnecessary when $p = 2$ and A and B are elementary.

(1.4) 2-Groups.

The following results on the derived length of G all assume G to be a finite 2-group. Proposition (1.3.1) is now extended to cover the case when the order of A_1 is p^2 and $p = 2$.

Proposition (1.4.1)

Let $G = AXB$ be a finite group with elementary abelian 2-subgroups A, X and B where X has order 2, $A^X = A$ and $B^X = B$. Let $A_1 = [A, X]$ and suppose that the order of A_1 is 4.

Then G has derived length at most 3.

Proof

We may assume that XB is non-abelian by Itô's result (see Notation). In particular we then have $A \cap X = 1 = B \cap X$, $A_1 \neq 1$ and

$B_1 = [B, X] = 1$.

Ito's result implies that $G/A_1{}^G$ and $G/B_1{}^G$ are both metabelian and it follows that

$$\frac{G}{A_1^G \cap B_1^G}$$

is also metabelian.

Thus to prove that the derived length of G is less than or equal to 3 it is sufficient to show that $A_1{}^G \cap B_1{}^G$ is abelian.

We begin by showing that we may assume that $A \cap B = 1$

As in the previous proposition suppose first that $A \cap B \neq 1$. Then there are 2 cases to consider.

Case i) $A_1 \cap B = 1$.
Case ii) $A_1 \cap B \neq 1$.

In case i) there exists a complement A^* to $A \cap B$ in A with $A_1 \leq A^*$. Then A^* is normalised by X and we can replace A by A^*. Thus we may assume that $A \cap B = 1$ in this case.

In case ii) we see that either $A_1 \leq B$ or $A_1 = (A_1 \cap B) \times A_2$ for some $A_2 \leq A_1$ with the order of A_2 equal to the order of $A_1 \cap B$ i.e. 2.

Since $A_1 \leq B$ implies that $A_1{}^G = A_1$ which is abelian we may assume that $A_1 \not\leq B$. Thus $A_1 = (A_1 \cap B) \times A_2$. Recall that $[A_1, B] = [A_2, B]$ is an elementary abelian 2-subgroup of G by (1.2.4).

Let $1 \neq a_2 \in A_2$ then $\langle a_2, b \rangle$ is dihedral for all $1 \neq b \in B$ since a_2 and b both have order 2. Then $[a_2, b]$ has order less than or equal to 2 because $[a_2, b] \in [A_1, B]$ and so a_2 centralises $[a_2, b]$ for all $b \in B$.

It follows that $A_2[A_2, B]$ is abelian since $A_2 = \langle a_2 \rangle$. Then

$$\begin{aligned} A_1{}^G &= A_1{}^B \\ &= A_1[A_1, B] \\ &= (A_1 \cap B)A_2[A_2, B] \end{aligned}$$

and so $A_1{}^G$ is abelian if and only if $[[A_2, B], A_1 \cap B] = 1$.

Clearly

$$[B, A_1 \cap B, A_2] = 1 = [A_1 \cap B, A_2, B]$$

and thus

$[A_2, B, A_1 \cap B] = 1$ by the Three Subgroup Lemma (see Notation). Hence G has derived length at most 3 in this case.

We have now reduced the problem to the case when $A \cap B = 1$ so consider this case.

Let $X = \langle x \rangle$.

We may assume that $A \cap XB = 1$ because if $1 \neq a = xb$ for some $a \in A$, $b \in B$ then $X = \{1, x\} \subseteq AB$ and so $G = AB$ is metabelian.

Recall that it is sufficient to prove that $A_1{}^G \cap B_1{}^G$ is abelian.

Suppose that

there exists $A_0 \leq C_A(X)$ with $A_1 \leq A_0XB \leq G$. $\quad (*)$

Then $L = A_0XB$ is metabelian by Itô (see Notation) and so $A_1 \leq L$ implies that $[A_1, B]B_1$ is abelian.

It follows that $A_1{}^G \cap B_1{}^G = A_1{}^B \cap B_1{}^A$

$$\leq (A_1 \cap B_1[B_1, A][A_1, B])[A_1, B]$$

and that this is abelian since $[B_1, A][A_1, B]$ is abelian by Lemma (1.2.3).

Thus if $(*)$ holds then G has derived length of at most 3 as required. We bear this fact in mind throughout the remainder of the proof.

Write $A = \langle a_1\rangle\times\langle a_2\rangle\times\ldots\times\langle a_n\rangle$.

X acts on A by conjugation and $|A_1| = 4$ so we may assume that A has the following decomposition into indecomposable X-submodules (see Introduction for more details).

$A = \langle a_1, a_2\rangle\times\langle a_3, a_4\rangle\times\langle a_5\rangle\times\langle a_6\rangle\times\ldots\times\langle a_n\rangle$,
where $a_1^x = a_1a_2$, $a_2^x = a_2$, $a_3^x = a_3a_4$, $a_i^x = a_i$ for $4 \le i \le n$.

G is nilpotent, being a product of the finite 2-groups AX and B, and so satisfies the normaliser condition. Hence we can construct a chain of subgroups of G, each one being normalised by the next subgroup in the chain, as follows :-

$XB \lhd \langle a_1'\rangle XB \lhd \langle a_1', a_2'\rangle XB \lhd \ldots \lhd \langle a_1', a_2', \ldots, a_n'\rangle XB = G$,
where $1 \ne a_i' \in A$ and $a_i' \notin \langle a_1', a_2', \ldots, a_{i-1}'\rangle$ for $1 \le i \le n$.

Let k be minimal such that $[a_k', x] \ne 1$ and let

$$A_3 = \langle a_1', a_2', \ldots, a_{k-1}'\rangle.$$

Then $[A_3, X] = 1$ and $\langle a_k', x\rangle \le A_3XB$. Without loss of generality we may assume that $a_k' = a_3$, and so $a_4 \in A_3XB$.

Then we can assume that $a_2 \notin A_3XB$ otherwise (*) holds and the result follows.

Now let l be minimal such that $[a_l', x] \notin \langle a_4\rangle$. Clearly $l > k$. Also let

$$A_4 = \langle a_1', a_2', \ldots, a_{l-1}'\rangle.$$

Then $\langle a_l', x\rangle \le A_4XB$.

Without loss of generality we may assume that $a_l' = a_1$. Then $a_2 \in A_4XB$ and $a_1 \notin A_4XB$.

Recall that $[a_2, B] \subseteq AB$ by Lemma (1.2.1). Hence

$$[a_2, B] \subseteq AB \cap A_4XB$$
$$= A_4\ (A \cap XB)B \quad \text{by Dedekind (see Notation)}$$
$$= A_4B$$

since $A \cap XB = 1$.

We show that (*) holds if $[a_2, B] \subseteq C_A(X)B$ and so we may assume that

$$a_3ab \in [a_2, B]$$

for some $a \in C_A(X)$ and $b \in B$ because $a_1 \notin A_4XB$.

Suppose that $[a_2, B] \subseteq C_A(X)B$.

Then

$$[a_2, B]B = [a_2, B]B \cap C_A(X)B$$
$$= ([a_2, B]B \cap C_A(X))B$$

by Dedekind's intersection lemma (see Notation), and this is a subgroup of G since $[\langle a_2\rangle, B]$ is normalised by B.

Now a_2 normalises $[\langle a_2\rangle, B] = [a_2, B]$ and so a_2 normalises

$$([\langle a_2\rangle, B]B \cap C_A(X))B.$$

It follows that $\langle a_2\rangle([a_2, B]B \cap C_A(X))A_3XB$ is a subgroup of G, because A_3 permutes with XB and with every subgroup of A.

Thus (*) holds because $a_4 \in A_3XB$ and

$$\langle a_2\rangle([a_2, B]B \cap C_A(X))A_3 \text{ is centralised by } x.$$

Therefore we may assume that $[a_2, B] \not\subseteq C_A(X)B$.

Then $1 \neq a_3ab \in [a_2, B]$ for some $a \in C_A(X)$ and $b \in B$, since

$$[a_2, B] \leq A_4XB$$

and $a_1 \notin A_4XB$.

$[a_2, B]$ is clearly normalised by XB, and since $a_2 \in C_A(X)$ we see that $[a_2, b]^a = [a_2, ba] = [a_2, a'x'b'] = [a_2, b'] \in [a_2, B]$ for all $a \in A$, $b \in B$ and some $a' \in A$, $b' \in B$ and $x' \in X$. So $[a_2, B]$ is normal in G and hence

$$[a_2, B] \ni [a_3ab, x] = a_4^b[b, x].$$

Thus $a_4b' \in [a_2, B]$ for some $b' \in B$.

Now, $\langle a_2, b\rangle$ and $\langle a_4, b\rangle$ are dihedral for all $b \in B$ and this implies that

$$[a_2, [a_2, b]] = 1 = [a_4, [a_4, b]]$$

since $[A_1, B]$ has exponent less than or equal to 2 by Corollary (1.2.4).

It follows that

$$\langle a_2\rangle^G = \langle a_2\rangle^B$$

and

$$\langle a_4\rangle^G = \langle a_4\rangle^B$$

are both abelian, since $[A_1, B]$ is abelian by (1.2.2).

Thus $[a_2, B] \ni a_4b'$ implies that

$$1 = [a_2, a_4b'] = [a_2, b'].$$

Also

$$\begin{aligned} 1 &= [[a_2, b], a_4b'] \\ &= [a_2, b, b'][a_2, b, a_4]^{b'} \end{aligned}$$

for all $b \in B$.

This then implies

$$1 = [a_2, b, a_4]$$

since $[a_2, b'] = 1$, for all $b \in B$.

Hence $[a_2, B, a_4] = 1$ and then $[B, a_4, a_2] = 1$ by the Three Subgroup Lemma (see Notation).

Thus $\langle a_2, a_4\rangle^B = A_1^G$ is abelian, by Corollary (1.2.2).

So we have shown that $A_1^G \cap B_1^G$ is abelian in all cases. Hence the derived length of G is at most 3.

$\square$

The next result can be deduced immediately from Propositions (1.3.1) and (1.4.1).

Theorem (1.4.2)

Let $G = AXB$ be a group of order 2^k, where $k \leq 12$, with elementary abelian subgroups A, X and B. Let X have order 2 and suppose also that $A^X = A$, $B^X = B$ and $A \cap B = 1$.

Then the derived length of G is at most 3.

Proof

As in Proposition (1.4.1) we may assume that $A \cap XB = 1 = X \cap B$. Then $|G| = |A||XB| = |A||X||B|$ and it follows that $|A||B| = 2^{k-1} \leq 2^{11}$.

A and B both decompose into a direct product of indecomposable X-submodules under conjugation by X, each having rank 1 or 2, since A and B are elementary abelian 2-groups and $|X| = 2$ (see Introduction for more details).

Thus we see that $|A_1| \leq |A|^{\frac{1}{2}}$ and $|B_1| \leq |B|^{\frac{1}{2}}$. Hence

$$|A_1||B_1| \leq (|A||B|)^{\frac{1}{2}} \leq 2^{\frac{11}{2}}$$

and so we see that

$$|A_1||B_1| \leq 2^5.$$

Clearly we may assume that A_1 and B_1 are both non-trivial, otherwise G is metabelian by Itô (see Notation).

We consider the following two cases separately :-

case i) $|A_1| = 2$ and $|B_1| \leq 2^4$.
case ii) $|A_1| = 2^2$ and $|B_1| \leq 2^3$.

In case i) we can apply Proposition (1.3.1) to show that G has derived length at most 3.

In case ii) we apply Proposition (1.4.1) to prove that G has derived length less than or equal to 3.

$\square$

(1.5) More Technical Results.

I would like to acknowledge help from G. Busetto, A. Leeves and S. E. Stonehewer in proving the following results (1.5.1), (1.5.2) and (1.6.1).

Before we can extend Theorem (1.4.2) to deal with the case when G has order 2^{13} we prove two general results that we shall find useful.

We consider a finite group $G = AXB$ where A, X and B are abelian subgroups and $A^X = A$ and $B^X = B$. In Lemma (1.5.1) we show that the largest normal subgroup L of G contained in AB factorises as the product of a subgroup of A with a subgroup of B. Then we use Itô's method (see [Sc3] (13.3.2)) to show that $[A, B, L] = 1$ in Corollary (1.5.2).

Lemma (1.5.1)

Let $G = AXB$ be a finite group with abelian subgroups A, X and B where $A^X = A$ and $B^X = B$. Let

$$L = \bigcap_{g \in G} (AB)^g.$$

Then

$$L = (L \cap A)(L \cap B).$$

Proof

We begin by showing that L is indeed a subgroup of G. Note that since G is finite it is sufficient to prove that L is closed.

Let $l_1, l_2 \in L$, then for any $g \in G$ we can find $a_1, a_2 \in A$ and $b_1, b_2 \in B$ such that $l_1 = (a_1b_1)^g$ and $l_2 = (a_2b_2)^g$. Thus we see that

$$\begin{aligned} l_1l_2 = (a_1b_1)^g(a_2b_2)^g &= (a_3b_3)^{a_2^{-1}g}(a_2b_2)^g \\ &= (a_2a_3b_3b_2)^g \in L \end{aligned}$$

for some $a_3 \in A$ and $b_3 \in B$.

Let $M = AL \cap BL$, so $L \leq M$.

Using Dedekind's intersection lemma (see Notation) we see that

$$\begin{aligned} M &= AL \cap BL \cap AB \\ &= A(AL \cap B) \cap (A \cap BL)B \\ &= (A \cap BL)(A \cap B)(AL \cap B) \\ &= (A \cap BL)(AL \cap B). \end{aligned}$$

Note that $B \cap AL$ is normalised by XB.

Let $b = al \in B \cap AL$ where $b \in B$, $a \in A$ and $l \in L$. If $a' \in A$ then

$$b^{a'} = al^{a'} \in AL$$

since $L \lhd G$.

Also $(bl^{-1})^{a'} = a = bl^{-1}$ and so $b^{a'} = bl^{-1}l^{a'} \in BL$. Hence

$$(B \cap AL)^G \leq M$$

and similarly

$$(A \cap BL)^G \leq M.$$

Thus $M \lhd G$ and then $M \subseteq AB$ implies that $M \leq L$. But $L \leq M$ and so $L = M$.

$A \cap LB \le L$ implies that

$$A \cap LB = A \cap LB \cap L$$
$$= A \cap L.$$

Similarly $AL \cap B = L \cap B$ and it follows that

$$L = M$$
$$= (L \cap A)(L \cap B).$$

$\square$

Corollary (1.5.2)

Let $G = AXB$ be a finite group with abelian subgroups A, X and B, where $A^X = A$ and $B^X = B$. Let

$$L = \bigcap_{g \in G} (AB)^g.$$

Then

$$[A, B, L'] = 1.$$

Proof

$L = (L \cap A)(L \cap B)$ by Lemma (1.5.1) and so

$$L' = [L \cap A, L \cap B].$$

We use Itô's method (see [Sc3] (13.3.2)) to show that $[A, B, L'] = 1$.

Let $a \in A$, $b \in B$, $a_1 \in A \cap L$ and $b_1 \in B \cap L$. Then since L is normal in G we can write

$$a_1{}^b = b_{1'} a_{1'}$$

and

$$b_1{}^a = a_{1''} b_{1''}$$

for some $a_{1'}, a_{1''} \in L \cap A$ and $b_{1'}, b_{1''} \in L \cap B$.

Consider

$$[a_1, b_1]^{ba} = [a_1{}^b, b_1]^a$$

$$
\begin{aligned}
&= [b_1 a_{1'}, b_1]^a \\
&= [a_{1'}, b_1]^a \\
&= [a_{1'}, a_{1''} b_{1''}] \\
&= [a_{1'}, b_{1'}]
\end{aligned}
$$

and

$$
\begin{aligned}
[a_1, b_1]^{ab} &= [a_1, b_1{}^a]^b \\
&= [a_1, a_{1''} b_{1''}]^b \\
&= [a_1, b_{1''}]^b \\
&= [b_1 a_{1'}, b_{1''}] \\
&= [a_{1'}, b_{1''}].
\end{aligned}
$$

Then $[a_1, b_1]^{ab} = [a_1, b_1]^{ba}$, for all $a \in A$, $b \in B$, $a_1 \in A \cap L$ and $b_1 \in B \cap L$.

Hence $[A, B, L] = 1$, as required.

$\square$

(1.6) Extension of Theorem (1.4.2) to the case $|G| = 2^{13}$.

We now extend Theorem (1.4.2) to the case when G has order 2^{13}, by doing a case by case analysis of such groups.

Proposition (1.6.1)

Let $G = AXB$ be a group of order 2^{13} with elementary abelian 2-subgroups A, X and B. Let X have order 2 and suppose also that $A^X = A$, $B^X = B$ and $A \cap B = 1$.

Then the derived length of G is at most 3.

Proof

First notice that we may assume that $A \cap XB = 1$. If not then there exists $1 \neq a = xb$ for some $a \in A$, $b \in B$ and $1 \neq x \in X$. Then $X = \{1, x\} \subseteq AB$ and $G = AB$ is metabelian.

Similarly we may assume that $X \cap B = 1$.

Let $A_1 = [A, X]$ and $B_1 = [B, X]$. Then Propositions (1.3.1) and (1.4.1) imply that we may assume that $|A_1| \geq 2^3$ and $|B_1| \geq 2^3$. Thus $|A| \geq 2^6$ and $|B| \geq 2^6$.

$|G| = |AXB| = |A||XB| = |A||X||B|$ because $A \cap XB = 1 = X \cap B$. Then $|G| = 2^{13}$ implies that $|A||B| = 2^{12}$ and so we must have $|A| = |B| = 2^6$ and $|A_1| = |B_1| = 2^3$.

$A_1{}^G = A_1{}^B \lhd G$ and $B_1{}^G = B_1{}^A \lhd G$, thus $A_1{}^G B_1{}^G$ is a subgroup of G. Lemma (1.2.1) shows that $\langle A_1, B\rangle \subseteq AB$, and similarly $\langle B_1, A\rangle \subseteq AB$. It follows that $A_1{}^G B_1{}^G \subseteq AB$, since both $A_1{}^G$ and $B_1{}^G$ are normal in G.

Let $L = \bigcap_{g \in G} (AB)^g$, then it is easy to see that L is a normal subgroup of G. In fact L is the largest normal subgroup of G contained in AB. In particular $A_1{}^G B_1{}^G \leq L$, thus $A_1 \leq L \cap A$ and $B_1 \leq L \cap B$.

Suppose that $A_1 = L \cap A$.

Then $LXB = A_1XB$ is a subgroup of G and it follows that $[A_1, B]B_1$ is abelian by Itô (see Notation).

Thus

$$\begin{aligned} A_1^G \cap B_1^G &= A_1[A_1, B] \cap B_1[B_1, A] \\ &\le A_1[A_1, B] \cap B_1[B_1, A][A_1, B] \\ &= (A_1 \cap B_1[B_1, A][A_1, B])[A_1, B] \end{aligned}$$

and this is abelian by (1.2.3) (as in the proof of (1.3.1)).

It follows that G has derived length at most 3 since G/A_1^G and G/B_1^G are both metabelian by Itō (see Notation).

Hence we may assume that $A_1 < L \cap A$ and, similarly that $B_1 < L \cap B$.

Note that $L = (L \cap A)(L \cap B)$ is metabelian by Itō (see Notation).

Consider $\langle A, B\rangle \lhd G$. If $\langle A, B\rangle = AB$ then $L = AB$ and thus G/L is abelian and so G has derived length at most 3. It follows that we may assume that $G = \langle A, B\rangle$.

Suppose that $L \cap A = A$, than G/L is abelian and the derived length of G is at most 3 because L is metabelian. So we may assume that $A_1 < L \cap A < A$ and $B_1 < L \cap B < B$.

We now show that $XL/L = Z(G/L)$.

Let $1 \neq x \in X$, then if $g = ax'b$ for some $a \in A$, $x' \in X$ and $b \in B$ we have

$$\begin{aligned} (xL)^g &= (xL)^{ax'b} \\ &= (x^aL)^{x'b} \\ &= (x[x, a]L)^{x'b} \\ &= (xL)^{x'b} \quad \text{since } A_1 \le L \\ &= (xL)^b \\ &= x^bL \\ &= x[x, b]L \\ &= xL. \end{aligned}$$

It follows that $XL/L \leq Z(G/L)$.

Suppose now that $abL \in Z(G/L)$ for some $a \in A$ and $b \in B$.

Then $(abL)^a = abL$ implies that $baL = abL$ and so ab has order ≤ 2 modulo L.

Also $(abL)^g = (ab)^g L = abL$ for all $g \in G$ and thus we see that $\langle ab \rangle L$ is normal in G.

Since ab has order ≤ 2 modulo L we also see that $\langle ab \rangle L \subseteq AB$. This implies that $\langle ab \rangle L = L$ and hence $ab \in L$, because L is the largest normal subgroup of G contained in AB.

Thus $abL \in Z(G/L)$ for some $a \in A$ and $b \in B$ implies that $ab \in L$.

Hence we must have

$$XL/L = Z(G/L).$$

We now consider G/L and recall that AL/L and BL/L must have orders 2 or 2^2.

Let $\bar{G} = G/L$ and let bars denote the images modulo L. Then

$$\bar{G} = \bar{A}\bar{X}\bar{B}$$

and we have 3 cases to consider :-

i). $\bar{A} = \langle \bar{a} \rangle$ and $\bar{B} = \langle \bar{b} \rangle$.

ii). $\bar{A} = \langle \bar{a} \rangle$ and $\bar{B} = \langle \bar{b}, \bar{b}' \rangle$.

iii). $\bar{A} = \langle \bar{a}, \bar{a}' \rangle$ and $\bar{B} = \langle \bar{b}, \bar{b}' \rangle$,

where $a, a' \in A$, $b, b' \in B$ are such that $a, a', b, b' \notin L$ and $aL \neq a'L$, $bL \neq b'L$.

Note that $A \cap XB = 1 = X \cap B$ implies that $\bar{A} \cap \bar{X}\bar{B} = 1 = \bar{X} \cap \bar{B}$.

Case i). $\bar{G}$ has order 8 and so is nilpotent of class at most 2.

Thus $\bar{G}' \leq Z(\bar{G})$ and this implies that $G' \leq XL$, since $Z(\bar{G}) = \bar{X}$.

Then $XG' = XG' \cap XL = X(XG' \cap L)$ and so

$$
\begin{aligned}
G'' \leq (XG')' &= (X(XG' \cap L))' \\
&\leq \langle [X, XG' \cap L], (XG' \cap L)' \rangle^G \\
&\leq \langle [X, XG'], L' \rangle^G \\
&\leq ([X, G']L')^G
\end{aligned}
$$

since $G' \leq XL$.

$G' = [A, B]$ because $G = \langle A, B \rangle$ and so the Three Subgroup Lemma (see Notation) may be applied to show that

$$[G', X] \leq [B, X, A][A, X, B] \lhd G$$

since $[B_1, A] = [B, X, A]$ and $[A_1, B] = [A, X, B]$ are normal in G by (1.2.3).

Writing $L' = [L \cap A, L \cap B]$ it is easy to see that $L' \lhd G$, and so

$$([X, G']L')^G = [X, G']^G L' \leq [A_1, B][A, B_1]L'.$$

Then Lemmas (1.2.3) and (1.5.2) imply that $([X, G']L')^G$ is abelian. It follows that G'' is abelian since $G'' \leq ([X, G']L')^G$.

Case ii). $\bar{G} = \bar{A}\bar{X}\bar{B}$ where $|\bar{B}| = 4$ and $\bar{A} = \langle \bar{a} \rangle$ for some $a \in A$ with $a \notin L$.

Then $\bar{X} = Z(\bar{G})$ and so $\bar{X}\bar{B}$ is an elementary abelian 2-group normalised by $\bar{a}$. It follows that $\bar{G}$ is nilpotent of class at most 2 and then G'' is abelian as in case i).

It remains to consider

Case iii). $\bar{G} = \bar{A}\bar{X}\bar{B}$ where $\bar{A} = \langle \bar{a}, \bar{a}' \rangle$ and $\bar{B} = \langle \bar{b}, \bar{b}' \rangle$ for some $a, a' \in A$ and $b, b' \in B$ with $a, a', b, b' \notin L$ and $aL \neq a'L$, $bL \neq b'L$.

Let $X = \langle x \rangle$.

Suppose that $\bar{X}\bar{B} \lhd \bar{G}$, then $\bar{a}$ acts on $\bar{X}\bar{B}$ by conjugation and the fixed point subgroup has rank 2 because $\bar{X} = Z(\bar{G})$. Thus the fixed point subgroup of $\bar{a}$ in $\bar{X}\bar{B}$ intersects $\bar{B}$ and without loss of generality we may assume that $[\bar{a}, \bar{b}] = 1$.

Hence $[\bar{a}, \langle \bar{b} \rangle^{\bar{G}}] = 1$ and $\langle \bar{b} \rangle^{\bar{G}} \leq \bar{X}\bar{B} \cap C_{\bar{X}\bar{B}}(\bar{a})$. It follows that

$$\langle \bar{b} \rangle^{\bar{G}} = \langle \bar{x}, \bar{b} \rangle.$$

Then $1 \neq [\bar{a}', \bar{b}] \in \langle \bar{b}, \bar{x} \rangle$ implies that $[\bar{a}', \bar{b}] = \bar{x}$.

Similarly $\bar{a}'$ acts on $\bar{X}\bar{B}$ by conjugation and the fixed point subgroup intersects $\bar{B}$. This fixed point subgroup does not contain $\bar{b}$ and so without loss of generality we may assume that $[\bar{a}', \bar{b}'] = 1$.

Hence $[\bar{a}', \langle \bar{b}' \rangle^{\bar{G}}] = 1$ and it follows that $\langle \bar{b}' \rangle^{\bar{G}} = \langle \bar{b}', \bar{x} \rangle$. Then

$$1 \neq [\bar{a}, \bar{b}'] \in \langle \bar{b}', \bar{x} \rangle$$

implies that $[\bar{a}, \bar{b}'] = \bar{x}$.

Thus when $\bar{X}\bar{B} \lhd \bar{G}$ we see that $\bar{G}/\bar{X}$ is abelian and hence $G' \leq XL$. Then G'' is abelian as in case i).

So we may assume that $\bar{A}\bar{X} \ntrianglelefteq \bar{G}$ and $\bar{B}\bar{X} \ntrianglelefteq \bar{G}$.

$\bar{G}$ is nilpotent and so satisfies the normaliser condition. Then without loss of generality we may assume that $\bar{a}$ normalises $\bar{X}\bar{B}$.

The fixed point subgroup of $\bar{a}$ in $\bar{X}\bar{B}$ has rank 2 and so intersects $\bar{B}$, hence we may assume that $[\bar{a}, \bar{b}] = 1$.

Then $\langle \bar{x}, \bar{b} \rangle = Z(\langle \bar{a} \rangle \bar{X}\bar{B}) \trianglelefteq \bar{G}$ implies that $\langle \bar{b} \rangle^{\bar{G}} = \langle \bar{x}, \bar{b} \rangle$.

It follows that $[\bar{a}', \bar{b}] = \bar{x}$.

$\bar{a}'$ does not normalise $\bar{X}\bar{B}$ since $\bar{X}\bar{B} \ntrianglelefteq \bar{G}$, but $\bar{a}'$ does normalise $\langle \bar{a} \rangle \bar{X}\bar{B}$ by the normaliser condition in $\bar{G}$.

Thus $[\bar{a}', \bar{b}'] = \bar{b}'^{\alpha} \bar{b}^{\beta} \bar{x}^{\gamma} \bar{a}$, for some $\alpha, \beta, \gamma \in \{0,1\}$.

$\bar{G}$ is a product of 2 abelian subgroups of exponent 2 and so $\bar{G}'$ is elementary abelian (by Holt and Howlett [Ho&Ho]).

Hence $[\bar{a}', \bar{b}']$ has order ≤ 2 and $\langle \bar{a}', \bar{b}' \rangle$ is a dihedral group with centre containing $[\bar{a}', \bar{b}']$.

It follows that

$$1 = [\bar{a}', \bar{b}', \bar{b}'] = [\bar{b}'^{\alpha} \bar{b}^{\beta} \bar{x}^{\gamma} \bar{a}, \bar{b}'] = [\bar{a}, \bar{b}'].$$

Then $[\bar{a}, \bar{b}] = 1 = [\bar{a}, \bar{b}']$ implies that $\bar{a} \in Z(\bar{G}) = \bar{X}$. This is a contradiction since $\bar{a} \neq 1$ and $\bar{A} \cap \bar{X} = 1$ and so this case cannot occur.

Thus we must have G of derived length at most 3 in all cases.

□

Remark

In (1.4.1), (1.4.2) and (1.6.1) we use the assumption that $p = 2$ only in order to show, at crucial stages of the argument, that certain subgroups of the form $\langle [a, b], a \rangle$ are abelian where $a \in A$ and $b \in B$.

At these stages we obtain $[a, b]$ of order dividing 2 by Holt and Howlett [Ho&Ho] and then $a^2 = 1 = b^2$ implies that

$$\langle a, b \rangle = \langle a, ab \mid a^2 = 1 = (ab)^4, a(ab)a = ba \rangle$$

which is a dihedral group. The centre of this group contains $(ab)^2 = [a, b]$ and thus $\langle [a, b], a \rangle$ and $\langle [a, b], b \rangle$ are abelian.

The remainder of the argument, in each case, does not use the assumption that $p = 2$.

Chapter 2 :

Special cases of finite products of three abelian p-groups

(2.1) Introduction.

This chapter deals with some special cases when G is a finite p-group which is a product of 3 abelian subgroups A, X and B, X being cyclic.

We usually require X to normalize both A and B and to have subnormal defect 2 in at least one of AX and XB. In each case a bound for the derived length of G is obtained which does not involve the order of X.

We then assume that X has subnormal defect 2 in both AX *and* XB and either find a bound for the subnormal defect of X in G (independent of the order of X), or show that in order to calculate such a bound we may assume that X has order p.

(The general case when $G = AXB$ is a finite p-group with abelian subgroups A, X and B where $|X| = p$, $A^X = A$, $B^X = B$ and $X \lhd^m AX$, $X \lhd^m XB$ is dealt with in Chapter 3 (3.3.1). A bound for the subnormal defect of X in G is obtained that depends only on m, for all primes p (see the remark following Theorem (3.3.1)).)

(2.2) Products of elementary abelian p-groups.

We begin by assuming that A and B are elementary abelian p-groups and find it helpful to tackle the cases $p = 2$ and $p \neq 2$ separately. When $p \neq 2$ we are able to use the identity

$$\sum_{i=1}^{p-1} i = 0 \pmod p$$

and obtain a better bound for the derived length of G than when $p = 2$.

Lemma (2.2.1)

Let $G = AXB$ be a finite 2-group with subgroups A, X and B where A and B are elementary abelian and $<x> = X$ is cyclic. Suppose that $A^X = A$, $B^X = B$, $X \lhd^2 AX$ and $X \lhd^2 XB$. Then

i) $<x^2> \lhd G$

ii) G has derived length at most 5.

Note that i) implies that in order to calculate the subnormal defect of X in G we may assume that X has order 2.

Proof of (2.2.1)

Let X have order 2^n.

We prove i), and then show that G has derived length at most 4 when $n = 1$. Then ii) follows by consideration of $G/<x^2>$.

i) If $A \cap X = 1$, then $X \lhd^2 AX$ implies that $[A, X, X] \leq A \cap X = 1$. Hence $[a, x^2] = [a, x]^2[a, x, x] = [a, x]^2 = 1$, for all $a \in A$. So $x^2 \in Z(AX)$ if $A \cap X = 1$, then $<x^2> \lhd AX$ is clear.

Suppose $A \cap X \neq 1$, then $A \cap X = <x^{2^{n-1}}>$ and $x^{2^{n-1}} \in Z(AX)$, since X has order 2^n.

Modulo $<x^{2^{n-1}}>$, $A \cap X$ is trivial. Thus we may use the case when $A \cap X = 1$ to give

$$<x^2>/<x^{2^{n-1}}> \leq Z(AX/<x^{2^{n-1}}>)$$

and

$$<x^2>/<x^{2^{n-1}}> \lhd AX/<x^{2^{n-1}}>.$$

Then $<x^2> \lhd AX$ in this case also.

Similarly we see that $\langle x^2\rangle \lhd XB$ and thus $\langle x^2\rangle \lhd G$.

ii) It is sufficient to prove that $G/\langle x^2\rangle$ has derived length at most 4, and so we assume that $n = 1$.

Recall that

$$[A, X]^G = [A, X]^B \subseteq AB$$

by (1.2.1), when $n = 1$.

$G/[A, X]^B$ is metabelian by Itô (see Notation) and so

$$G'' \leq [A, X]^B \subseteq AB.$$

Then $AG'' = A(AG'' \cap B)$ is a metabelian subgroup of G, by Itô's result (see Notation).

It follows that G'' is metabelian and G has derived length at most 4 when $n = 1$.

Thus $G/\langle x^2\rangle$ has derived length at most 4 and the result follows.

$\square$

Remark 1

If G satisfies the hypotheses of (2.2.1) with $|X| = 4$ then G has derived length at most 4. This is because $\langle x^2\rangle$ has order 2 and so lies in the centre of G. Consideration of $G/\langle x^2\rangle$ then gives $G'' \subseteq AB\langle x^2\rangle$ by (1.2.1), as in the proof of (2.2.1)ii). This implies that G'' is metabelian since $B\langle x^2\rangle$ is abelian.

Remark 2

Similarly, if G satisfies the hypotheses of (2.2.1) and AX is nilpotent of class 2, then $[A, X, X] = 1$ implies that $x^2 \in Z(AX)$ (as in the proof of (2.2.1)). Thus $A\langle x^2\rangle$ is abelian. Consideration of $G/\langle x^2\rangle$ implies that

$$G'' \subseteq AB\langle x^2\rangle = (A\langle x^2\rangle)B$$

and G" is metabelian by Itô (see Notation).

Example

In this example we show that the condition that AX be nilpotent of class 2 is not redundant in Remark 2. i.e. There do exist groups $H = AX$ satisfying the conditions of Lemma (2.2.1) with H nilpotent of class 3.

Let

$H = \langle a, b, x \mid a^2 = 1 = b^2 = x^8, [a, b] = 1, [a, x] = b, [b, x] = x^4\rangle.$

Then

$$H = AX$$

where

$$A = \langle a, b, x^4\rangle$$

and

$$X = \langle x\rangle.$$

It is easy to check that H is nilpotent of class 3.

We now consider finite p-groups G where $p \neq 2$ is prime.

Lemma (2.2.2)

Let $G = AXB$ be a finite p-group with abelian subgroups A, X and B where p is an odd prime. Suppose that A and B are elementary and $X = \langle x\rangle$ is cyclic with $A^X = A$, $B^X = B$, $X \lhd^2 AX$ and $X \lhd^2 XB$. Then

i) $x^p \in Z(G)$

ii) G has derived length at most 4.

Note that i) implies that in order to calculate the subnormal defect of X in G we may assume that X has order p.

Proof of (2.2.2)

i) Note that $[A, X, X] \leq A \cap X \leq Z(AX)$, and so

$$[a, x, x^i] = [a, x, x]^i$$

for all $a \in A$ and $i \in \mathbb{Z}$.

Let $a \in A$ and $\alpha \in \mathbb{N}$, then

$$\begin{aligned}[a, x^\alpha] &= [a, x^{\alpha - 1}][a, x]^{x^{\alpha - 1}} \\ &= [a, x^{\alpha - 1}][a, x][a, x, x^{\alpha - 1}] \\ &= [a, x^{\alpha - 1}][a, x][a, x, x]^{\alpha - 1}.\end{aligned}$$

Since $[A, X] \leq A$ is abelian it follows that

$$[a, x^p] = [a, x]^p[a, x, x]^{(p - 1) + (p - 2) + ... + 2 + 1}$$

for all $a \in A$.

Then

$$1 + 2 + ... + (p - 2) + (p - 1) \equiv 0 \pmod p$$

implies that

$$[a, x^p] = [a, x]^p = 1$$

for all $a \in A$.

Thus $x^p \in Z(AX)$, and similarly $x^p \in Z(XB)$.

ii) Working modulo $\langle x^p \rangle$, we see that $G'' \subseteq AB\langle x^p \rangle$, since $G/[A, X]^B$ is metabelian and then applying (1.2.1) to $G/\langle x^p \rangle$.

Then

$$AG'' = A(AG'' \cap B\langle x^p \rangle)$$

and this is a metabelian subgroup of G by Itô (see Notation), since $\langle x^p \rangle \leq Z(G)$.

Hence the derived length of G is at most 4.

$\square$

(2.3) Products of three abelian groups, two of which are cyclic.

We now turn our attention to finite p-groups $G = AXB$ with abelian subgroups A, X and B where X and B are cyclic. We begin by considering the case when X has order p and XB is a nilpotent subgroup of class 2 and prove that G has derived length at most 3.

Proposition (2.3.1)

Let $G = AXB$ be a finite p-group with abelian subgroups A, X and B where $X = \langle x \rangle$ has order p and $\langle b \rangle = B$. Suppose also that XB is a nilpotent subgroup of class 2, then the derived length of G is at most 3.

Proof

If $AB = \langle A, B \rangle$ then $|G| = (|AB||X|)/(|AB \cap X|)$ implies $|G : AB| \leq p$ which implies that AB is normal in G. Hence $G' \leq AB$ and G has derived length at most 3 by Itô (see Notation).

So we may assume that $AB \subset \langle A, B \rangle$. Since X has order p this implies that $G = \langle A, B \rangle$.

G is nilpotent and so A is subnormal in G. Thus $A^G < G$.

Then

$$A^G = A^G \cap AXB = A(A^G \cap XB),$$ by Dedekind (see Notation), and this implies that

$$A^G \cap XB < XB.$$

XB is nilpotent of class 2 and so

$$[x^p, b] = [x, b^p] = 1.$$

It follows that every proper subgroup H of XB is abelian because $H < XB$ implies that

either $H \geq B$ and so $H = B$,

or $H \cap B < B$ and so $H \leq \langle xb^i \rangle \langle b^p \rangle$ for some integer i.

In particular $A^G \cap XB$ is abelian, and thus $A^G = A(A^G \cap XB)$ is metabelian by Itô (seeNotation).

$G = \langle A, B\rangle$ implies that $G = A^G B$ and hence G/A^G is abelian.

Thus G has derived length at most 3.

□

An immediate corollary of (2.3.1) is

Corollary (2.3.2)

Let $G = AXB$ be a finite p-group where A, X and B are abelian subgroups, $|X| = p$ and B is cyclic of order p^2. Suppose that XB is a subgroup of G, then G has derived length at most 3.

Proof

$|XB|$ divides $|X||B| = p^3$. Hence XB is nilpotent of class at most 2 and (2.3.1) applies.

□

We now allow both X and B to be general cyclic p-groups and show that the finite p-group $G = AXB$ has derived length at most 4 when A is an abelian subgroup, X normalises both A and B, and XB is nilpotent of class at most 2. When X has subnormal defect 2 in AX we give a bound for the subnormal defect of X in G.

Proposition (2.3.3)

Let $G = AXB$ be a finite p-group with abelian subgroup A and cyclic subgroups X and B. Suppose that $A^X = A$, $B^X = B$ and XB is nilpotent of class at most 2. Then

i) $G'' \subseteq AXB_1$, where $B_1 = [B, X]$ and G has derived length at most 4.

If $X \lhd^2 AX$, then

ii) $X \lhd^8 G$.

Proof

i) We use induction on $|G|$ to show that $G'' \subseteq AXB_1$. Then Itô's result (see Notation) implies that the derived length of G is at most 4.

Suppose that $G'' \not\subseteq AXB_1$ and let G be a counterexample of minimal order.

Then we may assume that $B_1 \neq 1$, otherwise XB is abelian and $G'' = 1 \subseteq AXB_1$ by Itô's result (see Notation).

Suppose $A \cap B \neq 1$. Then $|G/A \cap B| < |G|$ and so

$$\begin{aligned}(G/(A \cap B))'' &= G''(A \cap B)/(A \cap B) \\ &\subseteq AXB_1(A \cap B)/(A \cap B).\end{aligned}$$

Hence

$$G'' \subseteq AXB_1(A \cap B) = AXB_1$$

and this contradicts the choice of G.

Thus we may assume that $A \cap B = 1$. Let $1 \neq axb \in Z(G)$, where $a \in A$, $x \in X$ and $b \in B$.

Without loss of generality we may assume that axb has order p.

Let $x' \in X$, then $axb = (axb)^{x'} = a^{x'}xb^{x'}$.

This implies that

$$[x', a] = [x', b^{-1}] \in A \cap B,$$

since XB is nilpotent of class at most 2.

Thus $[a, x'] = 1 = [b, x']$ for all $x' \in X$. Then

$$1 = [a, axb] = [a, b].$$

Hence

$$(axb)^p = a^p x^p b^p = 1.$$

We have two cases to consider :-

a) $AX \cap B = 1$

b) $AX \cap B \neq 1$

Case a) $AX \cap B = 1$ implies that $b^p = (a^p x^p)^{-1} = 1$.

Thus $b \in \Omega_1(B) \leq B_1$, since B is cyclic and $B_1 \neq 1$.

Consideration of $G/\langle axb\rangle$ then gives $G'' \subseteq AXB_1$, since

$$|G/\langle axb\rangle| < |G|.$$

This contradicts our choice of G.

Case b) $AX \cap B \neq 1$ implies that

$$\begin{aligned} 1 \neq (AX \cap B)^G &= (AX \cap B)^{BAX} \\ &= (AX \cap B)^{AX} \\ &\leq AX. \end{aligned}$$

Then $|G/(AX \cap B)^G| < |G|$, and so $G'' \subseteq AXB_1(AX \cap B)^G$.

Thus $G'' \subseteq AXB_1$ in this case also.

Hence G cannot be a counterexample, and so $G'' \subseteq AXB_1$.

ii) Corollary 1 of Stonehewer [St1] states :

Let $G = HK$ *be a metabelian group with subgroups* H, K *and* X *where* $X \lhd^m H$ *and* $X \lhd^m K$. *Then*

$$X \lhd^{2(m+1)} G.$$

Note that

$$G/G'' = (AXG''/G'')(BXG''/G'')$$

is metabelian

Thus we can apply the above result to G/G'' to obtain

$$XG'' \lhd^6 G .$$

Since

$$AXG'' = AX(AXG'' \cap B_1)$$

and $X(AXG'' \cap B_1)$ is abelian we see that

$$X^{AXG''} = X^{AX(AXG'' \cap B_1)} = X^A.$$

Then $X \lhd^2 AX$ implies $X \lhd X^A \lhd AXG''$.

It follows that $X \lhd^8 G$ as required.

$\square$

Chapter 3 :

Subnormality in, and solubility of, finite products of three abelian groups

(3.1) Introduction.

In this chapter we study finite groups G of the form

$$G = AXB$$

where A, X and B are abelian subgroups of G.

If $A^X = A$, $B^X = B$ and X is subnormal in both AX and XB, then a bound for the derived length of G is found in terms of the highest power of a prime dividing the order of X. When X has order p (p prime) we assume fewer hypotheses, in particular if we also assume that G is a p-group then we only require A and B to be abelian.

When X is a p-group normalising A and B with subnormal defect m in both AX and XB a bound for the subnormal defect of X in G is obtained as a function of m, n, p and the derived length of G where $|X| = p^n$. In view of the fact that the derived length of G can be bounded in terms of n, we obtain a bound for the defect of X in G that depends only on m, n and p. In the case when X has order p, the defect of X in G is bounded by a function of m only, for all primes p. If m is fixed and $|X| = p^n$ then the subnormal defect of X in G can be bounded in terms of n for all primes p.

A similar result is obtained when X has order $p_1^{n_1} p_2^{n_2} \ldots p_r^{n_r}$ for distinct primes $p_1, p_2, \ldots, p_r$ and positive integers $n_1, n_2, \ldots, n_r$.

(3.2) Preliminary Results.

We begin by proving 3 preliminary results. The proof of the first is analogous to that used by Scott ([Sc3] (13.2.5)) when he proves the following result :-

> *If* $G = HK$ *is a finite group with subgroups* H *and* K *and* p *is any prime, then there exist* $P \in Syl_pH$ *and* $Q \in Syl_pK$ *with* $PQ \in Syl_pG$.

Theorem (3.2.1)

Let $G = HK$ be a finite soluble group with subgroups H and K. Then there are Hall π-subgroups H_π and K_π of H and K respectively such that $H_\pi K_\pi$ is a Hall π-subgroup of G for any set of primes π.

Proof

Let H_1, K_1 be Hall π-subgroups of H and K respectively. A result of P. Hall (see [Sc3] (9.3.10)) states :-

> *If* G *is a finite soluble group,* π *a set of primes and* A *a* π*-subgroup of* G. *Then*
>
> *i) there is a Hall* π*-subgroup of* G *containing* A
>
> *ii) any 2 Hall* π*-subgroups of* G *are conjugate.*

Thus $H_1 \leq G_1$ and $K_1{}^g \leq G_1$ for some $G_1 \in Hall_\pi(G)$ and $g \in G$.
Scott [Sc3] (13.2.4) states :-

> *If* $G = HK$ *is a group with subgroups* H, K *where* H *is conjugate to* A *and* K *is conjugate to* B, *then* $G = AB$ *and there exists* $y \in G$ *such that* $A = H^y$ *and* $B = K^y$.

Hence we see that $G = HK^g$ and $H = H^y$, $K = K^{gy}$ for some $y \in G$. ${H_1}^y$, ${K_1}^{gy}$ are Hall π-subgroups of H and K and ${G_1}^y$ is a Hall π-subgroup of G.

We claim that

$$ {H_1}^y {K_1}^{gy} = {G_1}^y. $$

Clearly ${H_1}^y {K_1}^{gy} \subseteq {G_1}^y$. The reverse inclusion is proved by order considerations.

Let $|H| = ms_\pi$, $|K| = nt_\pi$, $|H \cap K| = lr_\pi$ where $(m, s_\pi) = 1$, $(n, t_\pi) = 1$, $(l, r_\pi) = 1$ and s_π, t_π and r_π are π-numbers.

Then

$$ |G| = (mn/l)(s_\pi t_\pi / r_\pi) $$

and

$$ (mn/l,\ s_\pi t_\pi / r_\pi) = 1. $$

Thus

$$ \begin{aligned} |{G_1}^y| &= s_\pi t_\pi / r_\pi, \\ |{H_1}^y| &= s_\pi, \\ |{K_1}^{gy}| &= t_\pi \end{aligned} $$

and

$$ |{H_1}^y \cap {K_1}^{gy}| \leq r_\pi. $$

Hence

$$ |{H_1}^y {K_1}^{gy}| \geq |{G_1}^y| $$

and so

$$ {H_1}^y {K_1}^{gy} = {G_1}^y. $$

The lemma is then true with $H_\pi = {H_1}^y$ and $K_\pi = {K_1}^{gy}$.

$\square$

An immediate corollary that we find useful is

Corollary (3.2.2)

Let $G = HK$ be a finite soluble group with subgroups H and K and suppose that p divides $|G|$, where p is prime. Then

$$G = H_{p'}K_{p'}G_p$$

for any $G_p \in Syl_pG$ and for some Hall p'-subgroups $H_{p'}$, $K_{p'}$ of H and K respectively.

Proof

Since G is a finite soluble group we may write $G = G_{p'}G_p$ for any Hall p'-subgroup $G_{p'}$ of G and any $G_p \in Syl_pG$ (see the result of P. Hall used in the proof of (3.2.1)).

Lemma (3.2.1) can then be applied to give the result.

□

Another corollary of Lemma (3.2.1) shows that if G is the product of abelian p-subgroups A, X and B, where X is a p-group, then under certain conditions there exists a Sylow p-subgroup of G with a particularly nice structure.

Corollary (3.2.3)

Let $G = AXB$ be a finite group with abelian subgroups A, X and B where X is a p-group, $A^X = A$ and $B^X = B$. Suppose that X is subnormal in XB, then

$$A_pXB_p \in Syl_pG$$

where A_p and B_p are the (unique) Sylow p-subgroups of A and B respectively.

Proof

Let XB = K. Then Proposition (2.2.1) of [LS] states :-

> *Let $\mathfrak{X}$ be a class of groups. Then every join of subnormal $\mathfrak{X}$-subgroups is an $\mathfrak{X}$-subgroup if and only if every product of normal $\mathfrak{X}$-subgroups is an $\mathfrak{X}$-group.*

This together with Fitting's Theorem ([R3] (5.2.8)), implies that XB = K is nilpotent since X and B are both subnormal in K and G is finite.

It follows that G is soluble by Kegel-Wielandt (see [Sc3] (13.2.9)), which states:-

> *If a finite group is the product of two nilpotent subgroups then it is soluble.*

Then Lemma (3.2.1) implies that $A_pK_p \in Syl_pG$ for some $A_p \in Syl_pA$ and $K_p \in Syl_pK$.

We now express K_p as the product of a subgroup of X with a subgroup of B by noting a result of Hall ([Sc3] (9.3.4)) :-

> *If* G *is a group,* H *is a Hall π-subgroup of* G *and* L *is a subnormal subgroup of* G*, then* $L \cap H$ *is a Hall π-subgroup of* L *where π is a set of primes.*

Applying this to K with $\pi = \{p\}$ we see that $X \cap K_p \in Syl_pX$. This implies that $X \cap K_p = X$, since X is a p-group.

Similarly we obtain

$$K_p \cap B \in Syl_pB$$

since $B \lhd K$. Thus we may write

$$K_p \cap B = B_p$$

where B_p is the unique Sylow p-subgroup of G.

Then

$$\begin{aligned} K_p &= K_p \cap XB \\ &= X(K_p \cap B) \end{aligned}$$

by Dedekind's intersection lemma (see Notation), and it follows that

$$A_pXB_p \in Syl_pG.$$

$\square$

(3.3) Products AXB where $|X| = p$.

We now move on to the main results of the chapter beginning with consideration of a finite p-group $G = AXB$ with abelian subgroups A, X and B where X has order p. The result obtained is then extended to finite nilpotent groups $G = AXB$ and, finally, with some additional hypotheses, to finite groups.

We then consider finite groups $G = AXB$ where X has a more general structure, still assuming that A, X and B are abelian.

Theorem (3.3.1)

Let $G = AXB$ be a finite p-group with abelian subgroups A, X and B where $|X| = p$. Then

i) $G'' \subseteq AB$ and G has derived length at most 4.

Suppose in addition that $A^X = A$, $B^X = B$, $X \lhd^m AX$ and $X \lhd^m XB$. Then

ii) $X \lhd^{8m + 2 + \alpha p} G$, where α is the highest power of p dividing

$$h = 2^{t-1} \prod_{i=1}^{t-1} (2^i - 1)$$

with $t = 2^{m-1}$.

Proof

Suppose that $G'' \not\subseteq AB$ and let G be a counterexample of minimal order.

Then $G = \langle A, B\rangle$ otherwise $AB = \langle A, B\rangle$ has index in G dividing p, which forces AB to be normal in G and this implies that $G' \subseteq AB$ which is a contradiction.

Let $\langle x\rangle = X$. G is nilpotent and so has non-trivial centre. There are 2 cases to consider :-

a) $Z(G) \cap AB = 1$,

b) $Z(G) \cap AB \neq 1$.

Case a)

Without loss of generality we can find $axb \in Z(G)$ of order p, for some $a \in A$ and $b \in B$.

We claim that

$$G = A\langle axb\rangle B$$

and then by Itô's theorem (see Notation) $G'' = 1 \subseteq AB$.

Suppose that $G \neq A\langle axb\rangle B$, then we can find $i, j \in \mathbb{N}$ with $1 \leq i < j \leq p-1$ and

$$(axb)^i = a'x^{\alpha}b',$$

$$(axb)^j = a''x^{\alpha}b''$$

for some $a', a'' \in A$, $b', b'' \in B$ and some α with $0 < \alpha \leq p-1$.

Then

$$[a'x^{\alpha}, B] = 1$$

and

$$[a''x^{\alpha}, B] = 1$$

and it follows that

$$[a'(a'')^{-1}, B] = 1.$$

This implies that

$$a'(a'')^{-1} \in Z(G)$$

since $G = \langle A, B\rangle$. Hence $a' = a''$, and similarly $b' = b''$.

Thus

$$(axb)^i = (axb)^j$$

and this is a contradiction since $1 \leq i < j \leq p-1$ and axb has order p.

So the claim must be true , and then $G'' = 1 \subseteq AB$. This contradicts our choice of G.

Case b)

$Z(G) \cap AB \neq 1$ and so $1 \neq ab \in Z(G)$ for some $a \in A$ and $b \in B$. Note that $(ab)^i = a^i b^i$ for all $i \in \mathbb{Z}$ and so $\langle ab\rangle \subseteq AB$.

Now $G'' \subseteq AB\langle ab\rangle$, since $|G/\langle ab\rangle| < G$. Hence

$$G'' \subseteq A\langle ab\rangle B = AB.$$

This contradicts our choice of G.

Thus G cannot be a counterexample, and so $G'' \subseteq AB$. Then $AG'' = A(AG'' \cap B)$ is a metabelian subgroup of G by Itô (see Notation).

It is now clear that the derived length of G is at most 4. This completes the proof of i).

ii) Corollary 1 of [St1] states :-

Let $G = HK$ *be a metabelian group with subgroups* X, H *and* K *where* $X \lhd^m H$ *and* $X \lhd^m K$. *Then*

$$X \lhd^{2(m+1)} G.$$

We apply this result to G/G'' to obtain

$$XG'' \lhd^{2(m+1)} G.$$

Hence it is sufficient to prove that

$$X \lhd^{6m + \alpha p} G''X.$$

Let $G_1 = AG''X$ and $J = AG''$. Then J is metabelian and normal in $G_1 = AX(G_1 \cap B)$.

Lemma 1 of [St1] says :-

Let $G = HK = AX$ *be a group with subgroups* H, K, A *and* X *where* A *is abelian and normal in* G. *If* $X \lhd^m H$ *and* $X \lhd^m K$, *then* $X \lhd^{2m} G$.

Applying this lemma to

$$\begin{aligned} JX/J' &= (J/J')(J'X/J') \\ &= (J'AX/J')(J'(AG''\cap B)X/J') \end{aligned}$$

we see that $J'X \lhd^{2m} JX = G_1$.

It remains to prove that $X \lhd^{4m + \alpha p} J'X$. Let

$$J_1 = J'A \cap J'B_0$$

where $B_0 = G_1 \cap B$. Then

$$J_1 = J'(A \cap J'B_0)$$
$$= J'(J'A \cap B_0),$$

by Dedekind's intersection lemma (see Notation). Also

$$(A \cap J'B_0)(J'A \cap B_0) = A(J'A \cap B_0) \cap J'B_0$$
$$= J'A \cap AB_0 \cap J'B_0$$
$$= J'A \cap J'B_0$$
$$= J_1$$

since $J_1 \subseteq AB_0$.

Thus we have

$$J_1 = J'A_1 = J'B_1 = A_1B_1,$$

where $A_1 = A \cap J'B_0$ and $B_1 = J'A \cap B_0$.

Let $N = (J' \cap A)(J' \cap B_0)$, then $N \lhd J_1$ and N is abelian since

$$J = A(AG'' \cap B)$$

is metabelian. Hence Lemma 1 of [St1], quoted above, may be applied once more to show that $X \lhd^{2m} NX$.

Working modulo N it is easy to see that

$$NX \lhd^{m} (A \cap B_0)NX$$

and so it remains to prove that

$$(A \cap B_0)NX \lhd^{m + \alpha p} J'X(A \cap B_0).$$

Note that

$$(A \cap B_0)N \lhd J_1X$$

and let bars denote images in J_1X modulo $(A \cap B_0)N$. Then

$$\bar{J}_1 = \bar{J}'\bar{A}_1 = \bar{J}'\bar{B}_1 = \bar{A}_1\bar{B}_1.$$

Also

$$J'(A \cap B_0)N \cap A_1(A \cap B_0)N = (J'N \cap A_1(A \cap B_0))(A \cap B_0)N$$

$$\begin{aligned} &= (J' \cap A_1)(A \cap B_0)N \\ &= (J' \cap A \cap J'B_0)(A \cap B_0)N \\ &= (J' \cap A)(A \cap B_0)N \\ &= (A \cap B_0)N \end{aligned}$$

and so $$\bar{J}' \cap \bar{A}_1 = 1.$$

Similarly $\bar{J}' \cap \bar{B}_1 = 1$ and $\bar{A}_1 \cap \bar{B}_1 = 1$. So $\bar{J}_1$ has the following structure :-

$$\bar{J}_1 = \bar{A}_1\bar{J}' = \bar{B}_1\bar{J}' = \bar{A}_1\bar{B}_1$$

where $\bar{A}_1$, $\bar{B}_1$, $\bar{J}'$ are abelian subgroups with

$$\bar{J}' \lhd \bar{J}_1$$

and

$$1 = \bar{A}_1 \cap \bar{J}' = \bar{B}_1 \cap \bar{J}' = \bar{A}_1 \cap \bar{B}_1.$$

Section 4 of [St1] shows that when a group $\bar{J}_1$ has this structure then it is possible to construct a commutative, associative ring R where the additive group of R is isomorphic to $\bar{J}'$ and the multiplicative group $1 + R$ is isomorphic to $\bar{A}_1$.

An automorphism x of $\bar{J}_1$ with x-invariant subgroups $\bar{A}_1$, $\bar{B}_1$, $\bar{J}'$ induces, in a natural way, an automorphism λ of R. Conversely the automorphism λ of R induces the automorphism x of $\bar{J}_1$, with x-invariant subgroups $\bar{A}_1$, $\bar{B}_1$, $\bar{J}'$.

The action of x on $\bar{A}_1$ (and on $\bar{B}_1$) is thus shown to be isomorphic to the action of λ on the multiplicative group $1 + R$ (with $(1 + r)^\lambda = 1 + r^\lambda$),

and the action of x on $\bar{J}'$ is isomorphic to the action of λ on the additive group of R. Since $1 + R$ is a multiplicative group, R is a radical ring (every element of the form $1 + r$ ($r \in R$) is a unit).

Lemma 5 of [St1], which states :-

> *Let* Y *be a group of automorphisms of the commutative radical ring* R *and suppose that* Y *stabilises a series of the multiplicative group* $1 + R$ *of length* m. *Then there is a positive integer* $h = h(m)$ *such that* Y *stabilises a series of the additive group* hR *of length* $\leq m$. *In fact with* $t = 2^{m-1}$,
>
> $$h = 2^{t-1} \prod_{i=1}^{t-1} (2^i - 1).$$

can be applied to the ring R associated with $\bar{J}'$.

It follows that

$$\bar{X} \lhd^m (\bar{J}')^h \bar{X},$$

since $X \leq \mathrm{Aut}\, G$ with $\bar{A}_1{}^X = \bar{A}_1$, $\bar{B}_1{}^X = \bar{B}_1$, $\bar{J}'^X = \bar{J}'$ and $\bar{X} \lhd^m \bar{A}_1\bar{X}$, $\bar{X} \lhd^m \bar{B}_1\bar{X}$, where

$$(\bar{J}')^h = \langle g^h \mid g \in \bar{J}' \rangle$$

and

$$h = 2^{t-1} \prod_{i=1}^{t-1} (2^i - 1)$$

with $t = 2^{m-1}$.

Then $(\bar{J}')^h$ is characteristic in $\bar{J}'$ and so $(\bar{J}')^h$ is normalised by $\bar{X}$.

It remains to show that

$$(\bar{J}')^h\bar{X} \lhd^{\alpha p} \bar{J}'\bar{X}.$$

$\bar{J}'/(\bar{J}')^h$ is an abelian p-group with exponent dividing h. Thus the exponent of $\bar{J}'/(\bar{J}')^h$ divides p^α, where $\alpha \in \mathbb{N}$ is such that $p^\alpha \mid h$ and $p^{\alpha+1} \nmid h$.

It is then easy to see that $(\bar{J}')^h\bar{X} \lhd^{\alpha p} \bar{J}'\bar{X}$, since $|X| = p$.

Hence

$$X \lhd^{2m} NX \lhd^m (A \cap B_0)NX \lhd^m (J')^h(A \cap B_0)NX \lhd^{\alpha p} (A \cap B_0)J'X \lhd^{2m} G_1$$

and

$$XG'' \leq G_1$$

with

$$XG'' \lhd^{2(m+1)} G$$

implies that

$$X \lhd^{8m+2+\alpha p} G.$$

$\square$

Remark

Clearly $\alpha p \leq p^\alpha \mid h$ and so $X \lhd^{8m+2+h} G$ and the subnormal defect of X in G can be bounded (poorly) by a function of m, since h is also a function of m.

An immediate corollary of Theorem (3.3.1) i) is

Proposition (3.3.2)

Let $G = AXB$ be a finite nilpotent group with abelian subgroups A, X and B where X has order p. Then G'' is a p-subgroup contained in AB, and G has derived length at most 4.

Proof

A and B are abelian and so we may write

$$A = A_p \times A_{p'}$$

and

$$B = B_p \times B_{p'},$$

where A_p, B_p are the Sylow p-subgroups of A and B respectively, and $A_{p'}$ and $B_{p'}$ are the Hall p'-subgroups of A and B.

Then

$$G = A_pA_{p'}XB_pB_{p'} = A_pXB_pA_{p'}B_{p'}$$

since elements of coprime order in a nilpotent group commute.

Let G_p be the Sylow p-subgroup of G, and $G_{p'}$ the Hall p'-subgroup of G.

Then

$$A_pXB_p \subseteq G_p$$

and

$$A_{p'}B_{p'} \subseteq G_{p'}.$$

Also we see that

$$G = A_pXB_pG_{p'} = G_pA_{p'}B_{p'}.$$

Thus

$$\begin{aligned} G_p &= G_p \cap A_pXB_pG_{p'} \\ &= A_pXB_p(G_p \cap G_{p'}) \\ &= A_pXB_p \end{aligned}$$

and similarly

$$G_{p'} = A_{p'}B_{p'}.$$

Hence

$$G = (A_pXB_p)\times(A_{p'}B_{p'})$$

and so

$$G'' = (A_pXB_p)''\times(A_{p'}B_{p'})'' = (A_pXB_p)''.$$

Theorem (3.3.1) i) now clearly implies that $G'' \subseteq A_pB_p$ and G has derived length at most 4.

$\square$

The following lemma shows that if $G = AXB$ is a finite group with abelian subgroups A, X and B where X is a p-group, $A^X = A$, $B^X = B$ and X is subnormal in both AX and XB, then in order to calculate the subnormal defect of X in G we may assume that G is a p-group.

Lemma(3.3.3)

Let $G = AXB$ be a finite group with abelian subgroups A, X and B where X is a p-group for some prime p, $A^X = A$ and $B^X = B$. Suppose that X is subnormal in AX and XB. Then provided that $l \geq 1$

$$X \triangleleft^l G \text{ if and only if } X \triangleleft^l A_pXB_p,$$

where A_p and B_p are the Sylow p-subgroups of A and B respectively.

(Note that $A_pXB_p \in Syl_p\, G$ by Corollary (3.2.3)).

Proof

Let $XB = K$. Then K is nilpotent by [LS] (2.2.1) and by Fitting's Theorem (as in the proof of (3.2.3)).

Then G is soluble by Kegel–Wielandt (see [Sc3] (13.2.9)), as in the proof of (3.2.3).

K is nilpotent and so

$$K = K_p \times K_{p'}$$

where K_p is the Sylow p-subgroup of K and $K_{p'}$ is the Hall p'-subgroup of K.

Since X is subnormal in K, [Sc3] (9.3.4) implies that

$$X \cap K_p \in Syl_p X.$$

Thus $X \le K_p$ and $[X, K_{p'}] = 1$.

Similarly $[H_{p'}, X] = 1$, where $H_{p'}$ is the Hall p'-subgroup of $H = AX$. Then by Corollary (3.2.2),

$$X^G = X^{G_p}$$

for any Sylow p-subgroup G_p of G.

Since A_pXB_p is a Sylow p-subgroup of G, by Corollary (3.2.3), we see that

$$X \lhd^{\mathrm{l}} G \text{ if and only if } X \lhd^{\mathrm{l}} A_pXB_p.$$

$\square$

We now remove the requirement for G to be nilpotent from Proposition (3.3.2) but assume that $B^X = B$ and that XB is a nilpotent subgroup of G, to ensure solubility of $G = AXB$. A slightly larger bound for the derived length of G is obtained in this case than in (3.3.1) and (3.3.2). However, using Lemma (3.3.3), we see that when $A^X = A$, $X \lhd^m AX$ and $X \lhd^m XB$, then the same bound for the subnormal defect of X in G is obtained as in Theorem (3.3.1) ii).

Theorem (3.3.4)

Let $G = AXB$ be a finite group with abelian subgroups A, X and B where X has order p (for some prime p), $B^X = B$ and XB is nilpotent. Then

i) $G''' \subseteq AB$ and the derived length of G is at most 5.

Suppose that in addition $A^X = A$, $X \lhd^m AX$ and $X \lhd^m XB$. Then

ii) $X \lhd^{8m+2+\alpha p} G$, where α is the highest power of p dividing

$$h = 2^{t-1} \prod_{i=1}^{t-1} (2^i - 1),$$

with $t = 2^{m-1}$.

Proof

i) A is abelian and XB is nilpotent, thus G is soluble by Kegel-Wielandt (see [Sc3] (13.2.9)).

Suppose that $G''' \not\subseteq AB$ and let G be a counterexample of minimal order.

Let N be a minimal normal subgroup of G. Since G is soluble, N is abelian.

There are 2 cases to consider:-

a) $N \subseteq AB$

b) $N \not\subseteq AB$.

Case a) implies that

$$G''' \subseteq ANBN = ANB = AB,$$

since $|G/N| < |G|$.

This is a contradiction of our choice of G.

Case b) implies that

$$1 \neq axb \in N$$

for some $a \in A$, $b \in B$ and $1 \neq x \in X$.

We claim that $G = ANB$.

Clearly $AN \ni xb$ and AN is a subgroup of G. Thus

$$(xb)^k \in AN$$

for all integers k.

Then $B^X = B$ implies that $(xb)^k = x^k b_k$ for some $b_k \in B$ and for each $k \in \mathbb{Z}$.

Hence $x^k \in ANB$ for all $k \in \mathbb{Z}$.

It follows that $AXB \subseteq ANB \subseteq G = AXB$, and the claim is true.

N is normal in G and abelian, this implies that $G'' \leq N$ by Itô (see Notation). Then

$$G''' = 1 \subseteq AB.$$

This is a contradiction of our choice of G.

Thus G cannot be a counterexample and so $G''' \subseteq AB$. Then $G^{(5)} = 1$, as in (3.3.1) i).

ii) By Lemma (3.3.3) provided that $l \geq 1$, we have

$$X \lhd^{l} G \text{ if and only if } X \lhd^{l} A_pXB_p$$

where A_p, B_p are the Sylow p-subgroups of A, B respectively and $A_pXB_p \in Syl_pG$ by (3.2.3).

Then A_p is characteristic in $A \lhd AX$ implies that $A_p{}^X = A_p$. Similarly $B_p{}^X = B_p$.

Theorem (3.3.1) ii) can now be applied to the p-group A_pXB_p to give

$$X \lhd^{8m+2+\alpha p} A_pXB_p,$$

where α is the highest power of p which divides

$$h = 2^{t-1} \prod_{i=1}^{t-1} (2^i - 1),$$

with $t = 2^{m-1}$.

It follows that $X \lhd^{8m+2+\alpha p} G$.

$\square$

Remark

$\alpha p \leq p^{\alpha} \leq h$ and so the subnormal defect of X in G can be bounded by a function of m only.

(3.4) Products AXB where $|X| = p^n$.

We continue to consider finite groups $G = AXB$ with abelian subgroups A, X and B where $A^X = A$ and $B^X = B$, but now allow X to have order p^n for some prime p and positive integer n.

The following theorem gives a bound for the derived length of G in terms of n. When $X \lhd^m AX$ and $X \lhd^m XB$ a bound for the subnormal defect of X in G is obtained as a function of m, n and p. If m is fixed it is noted that the defect of X in G can be bounded in terms of n only, for all primes p.

Theorem (3.4.1)

Let $G = AXB$ be a finite nilpotent group with abelian subgroups A, X and B where $A^X = A$, $B^X = B$ and X has order p^n, for some prime p and positive integer n. Then

i) $G^{(2n)} \subseteq AB$ and G has derived length at most $2(n + 1)$.

Suppose also that $X \lhd^m AX$ and $X \lhd^m XB$. Let $t = 2^{m-1}$ and

$$h = 2^{t-1} \prod_{i=1}^{t-1} (2^i - 1).$$

Then

ii) $X \lhd^{(e - 2)(4m + p^n\alpha) + 2(m + 1)} G$, where $\alpha \in \mathbb{N} \cup \{0\}$ is such that $p^\alpha \mid h$ but $p^{\alpha + 1} \nmid h$, d is the derived length of A_pXB_p (where A_p, B_p are the Sylow p-subgroups of A, B respectively) and $e = \max\{d, 2\}$.

Proof

i) We use induction on n.

The result is true when $n = 1$ by (3.3.1) i).

Suppose that the result is false and choose n minimal such that a counterexample exists. Let G be a counterexample of minimal order for this n.

G is nilpotent and so has non-trivial centre.

There are two cases to consider :-

a) $Z(G) \cap AB = 1$,

b) $Z(G) \cap AB \neq 1$.

Case a) There exists $1 \neq axb \in Z(G)$ for some $a \in A$, $b \in B$ and $x \in X \setminus AB$.

Then $1 = [axb, B] = [ax, B]$ implies that $\langle ax\rangle^G \leq AX$. Hence we obtain a minimal normal subgroup $N = \langle a_1x_1\rangle$ of G, where $a_1 \in A$ and $x_1 \in X \setminus AB$.

$[A, x_1] = 1$ and so $\langle x_1\rangle^G = \langle x_1\rangle^B = \langle x_1\rangle[\langle x_1\rangle, B]$. Thus

$$x_1^{\alpha}b' \in Z(G)$$

for some $b' \in B$ and $\alpha \in \mathbb{N}$. Note that $x_1^{\alpha} \notin AB$.

Then $[x_1^{\alpha}, B] = 1$ and so $1 \neq \langle x_1^{\alpha}\rangle \leq Z(G)$.

Consideration of $G/\langle x_1^{\alpha}\rangle$ yields $G^{(2n-2)} \subseteq AB\langle x_1^{\alpha}\rangle$ by minimality of n.

Hence $AG^{(2n-2)} = A(AG^{(2n-2)} \cap B\langle x_1^{\alpha}\rangle)$, and this is metabelian by Itô (see Notation).

Thus $G^{(2n-2)}$ is metabelian and $G^{(2n)} = 1 \subseteq AB$. This contradicts our choice of G.

Case b) $Z(G) \cap AB \neq 1$ implies that there exists a minimal normal subgroup N of G with $N \subseteq AB$.

Then $G^{(2n)} \subseteq ANBN = ANB = AB$, since $|G/N| < |G|$.
This contradicts our choice of G.

Thus G cannot be a counterexample and so $G^{(2n)} \subseteq AB$. Consideration of $AG^{(2n)} = A(AG^{(2n)} \cap B)$ implies that $G^{(2n+2)} = 1$.

ii) We may assume that G is a p-group, by Lemma (3.3.3).
We use induction on the derived length, d, of G.

$$XG'' \triangleleft^{2(m+1)} G$$

by Corollary 1 of [St1], and so the result is true for $d \leq 2$.

Suppose that $d \geq 3$.

Let $D = G^{(d-1)}$, then $1 \neq D \triangleleft G$ and D is abelian. Since G/D has derived length $d(G/D) = d - 1$ it is sufficient to show that

$$X \triangleleft^{4m + p^{n\alpha}} DX.$$

Let $G_1 = ADX \cap BDX$.
Then

$$G_1 = (A \cap BDX)DX = (ADX \cap B)DX$$

by Dedekind's intersection lemma (see Notation).

Also

$$\begin{aligned}(A \cap BDX)X(ADX \cap B) &= (AX \cap BDX)(ADX \cap B)\\ &= AX(ADX \cap B) \cap BDX\\ &= ADX \cap AXB \cap BDX\\ &= ADX \cap BDX\\ &= G_1.\end{aligned}$$

Thus $G_1 = A_1XD = B_1XD = A_1XB_1$, where $A_1 = A \cap BDX$ and $B_1 = B \cap ADX$.

Let $H_0 = A_1D \cap A_1X$ and $K_0 = A_1D \cap B_1X$.

Then

$$H_0D = A_1D \cap A_1DX = A_1D$$

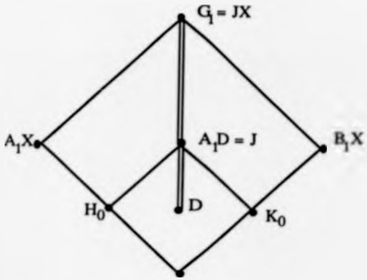
and

$$K_0D = A_1D \cap B_1XD = A_1D.$$

Also

$$\begin{aligned}H_0K_0 &= (A_1D \cap A_1X)(A_1D \cap B_1X)\\ &= A_1D \cap A_1X(A_1D \cap B_1X)\\ &= A_1D \cap A_1(A_1XD \cap B_1X) = A_1D.\end{aligned}$$

Let $J = A_1D$. Then $J = H_0K_0 = H_0D = K_0D$.



$H_0 \leq AX$ and $K_0 \leq BX$.

Thus Lemma 1 of [St1] implies that

$$X \lhd^{2m} (H_0 \cap D)(K_0 \cap D)X$$

since $D \lhd G$ is abelian.

$(H_0 \cap D)(K_0 \cap D)$ is normal in G_1 and H_0, K_0 are abelian modulo $(H_0 \cap D)(K_0 \cap D)$.

It follows that

$$N = (H_0 \cap K_0)(H_0 \cap D)(K_0 \cap D)$$

is normal in G_1.

$$(H_0 \cap D)(K_0 \cap D)X \lhd^m (H_0 \cap K_0)(H_0 \cap D)(K_0 \cap D)X,$$

since $H_0 \cap K_0 \leq AX$ and $X \lhd^m AX$.

Hence $X \lhd^{3m} NX$.

Let bars denote images in G_1 modulo N.

Then it remains to show that $\overline{X} \lhd^{m + p^{n\alpha}} \overline{G_1} = \overline{JX}$.

$$\begin{aligned} H_0N \cap K_0N &= H_0(K_0 \cap D) \cap K_0(H_0 \cap D) \\ &= (H_0(K_0 \cap D) \cap K_0)(H_0 \cap D) \\ &= (K_0 \cap D)(H_0 \cap K_0)(H_0 \cap D) = N. \end{aligned}$$

Similarly $H_0N \cap DN = N = K_0N \cap DN$ and so

$$\overline{J} = \overline{H}_0\overline{K}_0 = \overline{H}_0\overline{D} = \overline{K}_0\overline{D}$$

where $\overline{H}_0$, $\overline{K}_0$ and $\overline{D}$ are abelian $\overline{D} \lhd \overline{J}$ and

$$\overline{H}_0 \cap \overline{K}_0 = 1 = \overline{H}_0 \cap \overline{D} = \overline{K}_0 \cap \overline{D}.$$

As in the proof of (3.3.1)ii) we use section 4 of [St1] to show that there exists a commutative associative ring R such that the additive group of R is isomorphic to $\overline{D}$, and the multiplicative group $1 + R$ is isomorphic to $\overline{H}_0$ (and $\overline{K}_0$). It is shown that an automorphism x of $\overline{J}$ with x-invariant subgroups $\overline{H}_0$, $\overline{K}_0$, $\overline{D}$ induces an automorphism λ of R. Conversely the automorphism λ of R induces the automorphism x of $\overline{J}$.

The action of x on $\overline{H}_0$ (and on $\overline{K}_0$) is thus isomorphic to the action of λ on the multiplicative group $1 + R$, and the action of x on $\overline{D}$ is isomorphic to the action of λ on the additive group of R.

We apply Lemma 5 of [St1] to this ring R. Since $X \leq \mathrm{Aut}\, G$ with $\overline{H}_0^X = \overline{H}_0$, $\overline{K}_0^X = \overline{K}_0$, $\overline{D}^X = \overline{D}$ and

$$\overline{X} \lhd^m \overline{H_0X}$$

$$\overline{X} \lhd^m \overline{K_0}\overline{X}$$

it follows that

$$\overline{X} \lhd^m (\overline{D})^h\overline{X}$$

where

$$h = 2^{t-1} \prod_{i=1}^{t-1} (2^i - 1),$$

$t = 2^{m-1}$ and $(\overline{D})^h = \langle \overline{d}^h \mid \overline{d} \in \overline{D} \rangle$.

$\overline{D}$ is abelian and so $(\overline{D})^h$ is characteristic in $\overline{D} \lhd \overline{DX}$.

$\overline{D}/(\overline{D})^h$ is an abelian p-group with exponent dividing h. Hence the exponent of $\overline{D}/(\overline{D})^h$ divides p^α where $p^\alpha \mid h$, $p^{\alpha+1} \nmid h$ and $\alpha \in \mathbb{N} \cup \{0\}$.

It follows that $\overline{X}(\overline{D})^h \lhd^{p^n\alpha} \overline{DX}$, since $|\overline{X}| \leq p^n$.

Hence $X \lhd^{4m + p^n\alpha} DX$ and the result is clear if we take d to be the derived length of the Sylow p-subgroup A_pXB_p of G.

□

Remark

Note that if $p > 2^{t-1} - 1$ where $t = 2^{m-1}$, then $p \nmid h$ and so $\alpha = 0$ and the bound for the subnormal defect of X in G is independent of p. The derived length of G is bounded by a function of n when $|X| = p^n$ and so when m is fixed we can obtain a bound for the subnormal defect of X in G in terms of n, for all primes p.

The next result is a corollary of (3.2.3), (3.3.3) and (3.4.1).

Proposition (3.4.2)

Let $G = AXB$ be a finite group with abelian subgroups A, X and B where $|X| = p^n$ for some prime p and positive integer n. Suppose also that $A^X = A$, $B^X = B$ and X is subnormal in both AX and XB. Then

i) G has derived length at most $2(n + 2)$.

Suppose that, moreover, $X \lhd^m AX$ and $X \lhd^m XB$, then

ii) $$X \lhd^{(e-2)(4m+p^n\alpha)+2(m+1)} G$$

where α is defined as in (3.4.1)ii), d is the derived length of a Sylow p-subgroup of G and $e = \max\{d, 2\}$.

(In particular $d \leq 2(n + 1)$.)

Proof

It is sufficient to prove i), since ii) follows by (3.2.3), (3.3.3) and (3.4.1)ii) when G is soluble.

i) G is finite and so X is subnormal in G (Wielandt [Wi3]).

Since the product of normal p-subgroups of a finite group is a p-group, we can apply Proposition (2.2.1) of [LS] to show that X^G is a p-group (being generated by the subnormal p-subgroups X^g for all $g \in G$).

Then X^G lies in every Sylow p-subgroup of G.

Corollary (3.2.3) proves that A_pXB_p is a Sylow p-subgroup of G, where A_p, B_p are the Sylow p-subgroups of A, B respectively.

The derived length of A_pXB_p is at most $2(n + 1)$ by (3.4.1)i). Hence X^G has derived length less than or equal to $2(n + 1)$.

It follows that G has derived length at most $2(n + 2)$ since G/X^G is metabelian by Itô (see Notation).

$\square$

Remark (see also the remark following (3.4.1))

Note that if m is fixed then we can obtain a bound for the subnormal defect of X in G in terms of n alone.

(3.5) Products of three general abelian groups.

We conclude our study of finite groups G that are the product of 3 abelian subgroups A, X and B, with $A^X = A$, $B^X = B$ and X subnormal in both AX and XB, by considering a general X.

Letting X have order $p_1^{n_1}p_2^{n_2}...p_r^{n_r}$ for distinct primes $p_1,...,p_r$ and positive integers $n_1,...,n_r$ we obtain a bound for the derived length of G in terms of the greatest of $n_1,...,n_r$.

If we then suppose that $X \lhd^m AX$ and $X \lhd^m XB$ then a bound for the subnormal defect of X in G is obtained as a function of m, $\max_{i=1}^{r} \{n_i\}$ and $\max_{i=1}^{r} \{p_i^{n_i}\alpha_i\}$, where α_i is the highest power of p_i dividing

$$h = 2^{t-1} \prod_{i=1}^{t-1} (2^i - 1),$$

with $t = 2^{m-1}$.

Theorem (3.5.1)

Let $G = AXB$ be a finite group with abelian subgroups A, X and B where $|X| = p_1^{n_1} p_2^{n_2} \ldots p_r^{n_r}$ for distinct primes $p_1, \ldots, p_r$ and positive integers $n_1, \ldots, n_r$. Suppose also that $A^X = A$, $B^X = B$ and $X \lhd^m AX$ and $X \lhd^m XB$. Then

i) G has derived length at most $2(n + 2)$, where $n = \max_{i=1}^{r} \{n_i\}$.

ii) $$X \lhd^{(e-2)(4m + \alpha p^{n'}) + 2(m+1)} G$$

where d is the derived length of G, $e = \max \{d, 2\}$ and $\alpha p^{n'} = \max_{i=1}^{r} \{\alpha_i p_i^{n_i}\}$ (α_i is the largest power of p_i dividing

$$h = 2^{t-1} \prod_{i=1}^{t-1} (2^i - 1),$$

with $t = 2^{m-1}$)

Proof

G is finite and so X is subnormal in G (Wielandt [Wi3]).

Thus X^G is a nilpotent $\{p_1, p_2, ..., p_r\}$-subgroup, by [LS] (Proposition (2.2.1)).

Let $N = X^G$, then

$$N = N_{p_1} \times N_{p_2} \times ... \times N_{p_r}$$

where N_{p_i} is the Sylow p_i-subgroup of N for each i.

Now N_{p_i} is characteristic in $N \triangleleft G$ and so $N_{p_i} \triangleleft G$ for each i.

Let $N_{p_i'} = \langle N_{p_j} \mid j \neq i \rangle$, then $N_{p_i'}$ is normal in G and

$$XN_{p_i'}/N_{p_i'} \leq N/N_{p_i'}$$

is a p_i-subgroup for $i = 1, 2, ..., r$.

Proof of i).

(3.4.2)i) implies that

$$G^{(2n_i + 4)} \leq N_{p_i'}$$

for $i = 1, 2, ..., r$.

Thus

$$G^{(2n + 4)} \leq \bigcap_{i=1}^{r} N_{p_i'} = 1,$$

where $n = \max_{i=1}^{r} \{n_i\}$.

Proof of ii).

$$[G, {}_{m_i}X] \leq XN_{p_i'}.$$

where m_i is the subnormal defect of $XN_{p_i'}/N_{p_i'}$ in $G/N_{p_i'}$.

Then (3.4.2)ii) implies that

$$m_i \le (e - 2)(4m + \alpha_i p_i^{n_i}) + 2(m + 1).$$

It follows that

$$[G, {}_{\bar{m}}X] \le \bigcap_{i=1}^{r} XN_{p_i'},$$

where

$$\bar{m} = \max_{i=1}^{r} \{m_i\} \le (e - 2)(4m + \max_{i=1}^{r} \{\alpha_i p_i^{n_i}\}) + 2(m + 1).$$

Notice that

$$|XN_{p_i'} : X| = |N_{p_i'} : N_{p_i'} \cap X|$$

and this is a p_i'-number.

Thus

$$|\bigcap_{i=1}^{r} XN_{p_i'} : X|$$

is a p_i'-number for all i.

Hence

$$\bigcap_{i=1}^{r} XN_{p_i'} = X$$

and so

$$[G, {}_{\bar{m}}X] \le X.$$

The subnormal defect of X in G is then at most $\bar{m}$.

$\square$

Remark

If $n_1 = n_2 = \ldots = n_r = 1$ in (3.5.1), then (3.3.4) implies that G has derived length at most 5 and the defect of X in G is less than or equal to $8m + 2 + \alpha p$, where $\alpha p = \max_{i=1}^{r} \{\alpha_i p_i\}$. In particular $\alpha_i p_i \le p_i^{\alpha_i} \mid h$ for each i and so $X \triangleleft^{8m+2+h} G$ and the defect of X in G can be bounded (poorly) by a function of m, since h depends only upon m.

Chapter 4 :
Infinite Groups

(4.1) Introduction.

In this chapter we study infinite groups, in particular groups G of the form

$$G = AXB$$

where A, X and B are abelian p-subgroups of G with finite exponent and X is cyclic. We also require X to normalise both A and B and to have subnormal defect 2 in both AX and XB.

The aim is to find a bound for the derived length d(G) of G in terms of n, where $X \cong C_{p^n}$. Under certain additional conditions it has also been possible to obtain a bound for d(G) that is independent of the order of X.

Most of the results of this chapter appear as corollaries to Proposition (4.2.3) and Theorem (4.3.1), using results obtained in Chapters 2 and 3 for finite groups. In Theorem (4.3.1) we show that all the groups studied in this chapter are locally finite p-groups and Proposition (4.2.3) then demonstrates how the results proved in earlier chapters can be applied.

(4.2) Four useful results.

First we prove a preliminary lemma the proof of which is analogous to that used, in [Sc3] (13.2.7), to prove :-

> *Let* $G = HK$ *be a finite group with subgroups* H *and* K. *Suppose that* $J = \langle H_1, K_1 \rangle$, *where* $H_1 \lhd H$ *and* $K_1 \lhd K$ *and let* $N = N_G(J)$. *Then* $N = (N \cap H)(N \cap K)$.

Lemma (4.2.1)

Let $G = HK$ be a periodic group with subgroups H and K. Suppose that $J = \langle H_1, K_1 \rangle$, where $H_1 \lhd H$ and $K_1 \lhd K$ and write $N = N_G(J)$.

Then $N = (N \cap H)(N \cap K)$.

Proof

Let $g \in N$, then $g = hk^{-1}$ for some $h \in H$ and $k^{-1} \in K$.
Since $J = J^{hk^{-1}}$ it follows that $J^h = J^k$ and thus

$$J^h \geq H_1{}^h = H_1$$

and

$$J^h = J^k \geq K_1{}^K = K_1.$$

Hence $J^h \geq J = \langle H_1, K_1 \rangle$.

Since G is periodic we see that $J^h \neq J$ leads to a contradiction ($J^h \supsetneq J$, therefore $J^{h^2} \supsetneq J^h \supsetneq J$ etc).

Thus $J^k = J^h = J$, $h \in N \cap H$ and $k \in N \cap K$. It follows that

$$N \subset (N \cap H)(N \cap K).$$

The reverse inclusion is clear.

$\square$

Definition

Let $G = AXB$ be a group with subgroups A, X and B, where $A^X = A$ and $B^X = B$.

Call a section H/N of G (i.e. $N \lhd H \leq G$) *compatibly factorisable (c.f.)* if there are subgroups A_1, X_1 and B_1 of A, X and B respectively such that

$$H/N = \left(\frac{A_1N}{N}\right)\left(\frac{X_1N}{N}\right)\left(\frac{B_1N}{N}\right).$$

where $(A_1N)^{X_1} = A_1N$ and $(B_1N)^{X_1} = B_1N$.

Proposition (4.2.2)

Let $G = AXB = \langle A, B \rangle$ be a locally finite group with subgroups A, X, B where $A^X = A$ and $B^X = B$. Suppose that X is finite and all finite c.f. sections of G have derived length less than or equal to d for some $d \in \mathbb{N}$.

Then G has derived length d(G) less than or equal to 1 + d.

Proof

Let J be a finite subgroup of G. Then there are finite subgroups $A_1 \leq A$ and $B_1 \leq B$ such that $A_1 \lhd AX$, $B_1 \lhd BX$ and $J \leq \langle A_1, B_1 \rangle$.
Note that $J_1 = \langle A_1, B_1 \rangle$ is finite.

Let $N = N_G(J_1)$ so that (by (4.2.1))

$$\begin{aligned} N &= (N \cap AX)(N \cap XB) \\ &= (N \cap A)X(N \cap B) \end{aligned}$$

by Dedekind's intersection lemma (see Notation) (since $N \geq X$).

Let $C = C_G(J_1)$, then N/C is finite because it is isomorphic to a subgroup of Aut J_1. Notice that

$$N/C \geq J_1C/C \cong J_1/J_1 \cap C$$

and

$$J_1 \cap C = Z(J_1).$$

Hence $d(J_1) \leq 1 + d(N/C)$. But N/C is a finite c.f. section of G and hence $d(N/C) \leq d$.

It follows that G has derived length $d(G) \leq 1 + d$.

$\square$

An immediate corollary that we find useful in the proof of the main theorem is

Corollary (4.2.3)

Let $G = AXB$ be a locally finite group with subgroups A, X, B where $A^X = A$ and $B^X = B$. Suppose that X is finite and all finite c.f. sections of G have derived length less than or equal to d. Then the derived length of G is at most $d + 2$.

Proof

Let $H = < A, B >$. Then by Dedekind's intersection lemma (see Notation)

$$H = H \cap AXB = A(H \cap XB) = A(H \cap X)B.$$

Hence H has derived length $d(H) \leq 1 + d$, by (4.2.2).

Then, since H is normal in G and $G/H \cong X/(X \cap H)$, we see that G has derived length at most $2+d$.

$\square$

We now prove a general result about the action of a cyclic p-group on an elementary abelian p-group. This lemma is found to be very useful in the proof of Theorem (4.3.1).

Lemma (4.2.4)

Let B be an elementary abelian p-group and X a cyclic p-group such that $B^X = B$ and $B \cap X = 1$. Let $X = \langle x \rangle$, then B is finite if and only if the number of fixed points in B under conjugation by x is finite.

Proof

Suppose that the number of fixed points in B under conjugation by x is m and let X have order p^n.

Suppose B is infinite and let B_1 be a subgroup of B generated by mp^n+1 independent elements. Then $B_1{}^X = B_2$ is finite since XB is a locally finite p-group.

We can view B_2 as an X-module, the action of X being conjugation. Thus B_2 can be written as the direct product of indecomposable X-modules, each having rank at most p^n. However, this implies that B_2 contains at least m+1 points that are fixed under the action of x.

Thus we obtain a contradiction unless B is finite.

The reverse implication is clear.

$\square$

(4.3) Local finiteness of certain groups.

We are now in a position to prove the main theorem of Chapter 4, Theorem (4.3.1). This theorem shows that certain groups of the form AXB, where A, X, B are abelian p-groups of finite exponent, are locally finite (i.e. all finitely generated subgroups are finite) and so allows us to apply either (4.2.2) or (4.2.3) and some of the results of Chapters 2 and 3.

Theorem (4.3.1)

Let $G = AXB$ be a group with abelian p-subgroups A, X, B of finite exponent, where X is cyclic of order p^n. Suppose also that $A^X = A$, $B^X = B$, $X \triangleleft^2 AX$ and $X \triangleleft^2 XB$. Then G is a locally finite p-group.

Proof

A result of G.Baumslag ([R1] (6.34)) states that

A p-group G which is an extension of a nilpotent group of finite exponent by a finite group is nilpotent.

Thus we see that AX and XB are nilpotent. It follows that $A_0 = C_A(X)$ and $B_0 = C_B(X)$ are non-trivial since $A \lhd AX$ and $B \lhd BX$.

A subgroup which occurs in some ascending series of a group G is called an ascendant subgroup of G (see [R3] page 344).

The main content of the proof is the following claim.

Claim

Let $a_0 \in A_0$. Then $\langle a_0 \rangle$ is ascendant in G. (1)

Proof of claim

To prove the claim it is sufficient to show that

there exists $1 \neq N \lhd G$ with $\langle a_0 \rangle$ subnormal in $\langle a_0 \rangle N$ (2)

Transfinite induction then proves the claim.

We may assume that $A \cap B = 1$, otherwise taking $N = A \cap B$ in (2) gives $\langle a_0 \rangle \lhd \langle a_0 \rangle N$. Notice that if $A \cap B = 1$, then $C_G(X) = A_0XB_0$.

Suppose that

there exists $1 \neq L \lhd G$, for some locally finite p-group L (3)

Then $$AXL = AX(AXL \cap B)$$

is a locally finite p-group and so is soluble by Corollary (4.2.3) and

Theorem (3.4.1).Hence L has an elementary abelian characteristic subgroup $N \neq 1$. $N \lhd G$ and so a_0 acts on N by conjugation and $\langle a_0\rangle N$ is nilpotent of class less than or equal to $p^m = |a_0|$.

It follows that $\langle a_0\rangle$ is subnormal in $\langle a_0\rangle N$.

So to prove (2) it is sufficient to establish (3).

To show the existence of L in (3) we consider two cases:

Case i) There is some non-trivial $\hat{a}_0 \in A_0 \cap [A_0, B_0]$.
Case ii) $A_0 \cap [A_0, B_0] = 1$.

In case i) we show that (3) holds with $L = \langle\hat{a}_0\rangle^G$ and in case ii) with $L = G$.

Case i) There is some non-trivial $\hat{a}_0 \in A_0 \cap [A_0, B_0]$.

We show that (2) holds for $\langle\hat{a}_0\rangle$, hence by transfinite induction $\langle\hat{a}_0\rangle$ is ascendant in G.

Let $\mathfrak{X}$ be a class of groups, then $L\mathfrak{X}$ is defined to be the class of groups G such that every finitely generated subgroup of G is contained in an $\mathfrak{X}$-group. If $\mathfrak{X} = L\mathfrak{X}$ then $\mathfrak{X}$ is said to be L-closed. It is shown in (10) on page 20 of [R1], that any L-closed class $\mathfrak{X}$ of groups in which the product of any two normal $\mathfrak{X}$-subgroups is itself an $\mathfrak{X}$-group, is closed under the formation of the join of ascendant $\mathfrak{X}$-subgroups. Then $1 \neq \langle\hat{a}_0\rangle^G$ is a locally finite p-group, being generated by ascendant cyclic p-subgroups. Hence (3) holds with $L = \langle\hat{a}_0\rangle^G$,if $\langle\hat{a}_0\rangle$ is ascendant in G.

We may assume that $C_B(\hat{a}_0) = 1$ since $1 \neq b \in C_B(\hat{a}_0)$ implies that

$$1 = [\langle \hat{a}_0 \rangle, \langle b \rangle^{AX}].$$

Then (2) is true for $\langle \hat{a}_0 \rangle$ with $1 \neq N = \langle b \rangle^{AX} \lhd G$.

We may also assume that $A \cap X = 1$, since $(A \cap X)^G$ is a locally finite p-group.

It follows that $[A,X] \leq A_0$.

Recall that $C_G(X) = A_0XB_0$ which is metabelian by Itô (see Notation). Hence $[A_0, B_0]$ is abelian and thus

$$1 = [\hat{a}_0, [A_0, B_0]].$$

Every element of $[A_0, B_0]$ has the form $a_0'x'b_0'$ for some $a_0' \in A_0$, $b_0' \in B_0$ and $x' \in X$. However

$$1 = [\hat{a}_0, a_0'x'b_0'] = [\hat{a}_0, b_0']$$

and $C_B(\hat{a}_0) = 1$. Hence $b_0' = 1$ and so $[A_0, B_0] \leq A_0X$.

It follows that B_0 normalises A_0X and thus $B_0{}^A$ normalises A_0X since $[X, A] \leq A_0$. Then setting $N = B_0{}^A$ we have

$$[N, \langle \hat{a}_0 \rangle, \langle \hat{a}_0 \rangle] = 1$$

and so (2) holds for $\langle \hat{a}_0 \rangle$, as required.

Case ii) $A_0 \cap [A_0, B_0] = 1$.

Let $C = C_G(X) = A_0XB_0$.

We may assume that $C_{B_0}(a_0) = 1$, otherwise taking $N = (C_{B_0}(a_0))^G$ we have $\langle a_0 \rangle \lhd \langle a_0 \rangle N$ and so (2) holds for $\langle a_0 \rangle$.

We consider the cases with C' finite and C' infinite separately.

If C' is finite then $|C : C_C(a_0)|$ is finite since a_0 has only a finite number of conjugates in C, and so $|B_0 : C_{B_0}(a_0)|$ is finite. Hence B_0 is finite. B is a direct product of cyclic p-groups and we can apply Lemma (4.2.4) to show that the subgroup generated by all elements of B with order p is finite. It follows that B is finite because it has finite exponent.

Therefore $|G : AX|$ is finite and this implies that $|G : \mathrm{Core}_G(AX)|$ is finite, where $\mathrm{Core}_G(AX) = \bigcap_{g \in G} (AX)^g$.

Hence G is a locally finite p-group and (3) holds with L = G.

Now consider the case with C' infinite. We show that this possibility cannot happen.

Recall that $C = A_0XB_0$ is a product of two abelian p-groups of finite exponent. Thus C is a p-group of finite exponent,by Holt and Howlett [Ho&Ho] and C' is abelian. Hence C' is a direct product of cyclic p-groups.

Let $E = \Omega_1(C')$, the subgroup of C' generated by all elements of order p.

$$\begin{array}{l} C = A_0XB_0 \\ \Big| \\ C' = [A_0, B_0] \\ \Big| \\ E = \Omega_1(C') \\ \Big| \\ 1 \end{array}$$

Then E is infinite. E is characteristic in C' and hence normal in C, thus $a_0 \in C$ acts on E by conjugation.

The fixed points of E under the action of a_0 must lie in A_0X since $C_{B_0}(a_0) = 1$. Because we are in case ii) we have $A_0 \cap [A_0, B_0] = 1$ and hence $A_0 \cap E = 1$, since $C' = [A_0, B_0]$. It follows that all the non-trivial fixed points must involve some power of x, where $X = \langle x \rangle$.

There are infinitely many fixed points, by Lemma (4.2.4). Therefore there must be some α such that $a_0'x^{\alpha}$ and $a_0''x^{\alpha}$ are different fixed points in E for some $a_0', a_0'' \in A_0$.

Then $1 \neq (a_0'x^{\alpha})(a_0''x^{\alpha})^{-1}$ is in E and is fixed under conjugation by a_0. Thus $1 \neq (a_0')(a_0'')^{-1} \in E \cap A_0 = 1$. This is a contradiction and so C' is not infinite.

(3) has now been established in all cases.

It follows that $\langle a_0 \rangle$ is ascendant in G and the claim is proved.

By symmetry it can be seen that $\langle b_0 \rangle$ is ascendant in G for all $b_0 \in B_0$.

Hence $\langle A_0, B_0 \rangle^G$ is a locally finite p-group (being generated by ascendant cyclic p-subgroups (see [R1] p.20(10))).

We may assume that $A \cap X = 1 = X \cap B$ since $(A \cap X)^G$ and $(X \cap B)^G$ are locally finite groups. Then $[A, X] \leq A_0$ and $[B, X] \leq B_0$ and so AX and XB are abelian modulo $\langle A_0, B_0 \rangle^G$. Then $G/\langle A_0, B_0 \rangle^G$ is metabelian with finite exponent equal to a power of p, thus it is a locally finite p-group (periodic soluble groups are locally finite, see Robinson [R3] (5.4.11)).

It follows that G is a locally finite p-group as required.

$\square$

The requirement that X be subnormal in at most 2 steps in AX and XB plays a crucial part in the above argument and appears to be difficult to remove. The following corollaries all include this hypothesis (except for (4.4.4), where $X \lhd^2 AX$ and $X \lhd^2 XB$ may be deduced). Usually it is required only in the proof that G is a locally finite p-group.

(4.4) Applications of some results of Chapters 2 and 3 to infinite groups.

Corollary (4.4.1)

Let $G = AXB$ be a group with abelian p-subgroups A, X and B where $X \cong C_{p^n}$ and A and B have finite exponent. Suppose that $A^X = A$, $B^X = B$, $X \lhd^2 AX$ and $X \lhd^2 XB$. Then G has derived length $d(G) \leq 2n + 3$.

Proof

By Theorem (4.3.1) G is a locally finite p-group.

If $G = \langle A, B\rangle$, then G satisfies the hypothesis of Proposition (4.2.2) with $d = 2n + 2$, by (3.4.1). Hence $d(G) \leq 2n + 3$.

So suppose $G \neq \langle A, B\rangle$ and note $\langle A, B\rangle \lhd G$. Let $X = \langle x\rangle$, then

$$L = \langle A, B\rangle = A\langle x^{\alpha}\rangle B \text{ for some } \alpha \equiv 0 \bmod p.$$

L satisfies the hypotheses of Proposition (4.2.2) with $d = 2n$, by (3.4.1), because $|x^{\alpha}| < p^n$. Hence L has derived length at most $2n + 1$.

As $L \lhd G$ and G/L is abelian it follows that $d(G) \leq 2n+2$.

Thus G certainly has derived length less than or equal to $2n+3$ as required.

$\square$

The following results are special cases of Corollary (4.4.1). By adding further restrictions to the group G it is possible to obtain bounds for the derived length of G that do not depend upon the order of X. In adding further restictions we find it necessary to distinguish between the cases $p = 2$ and $p \neq 2$ for the prime p.

Corollary (4.4.2)

Let $G = AXB$ be a group with abelian p-subgroups A, X and B where $X \cong C_{p^n}$ and A and B have finite exponent. Suppose that $A^X = A$, $B^X = B$, $X \lhd^2 AX$ and $X \lhd^2 XB$ and in addition A is elementary and $p \neq 2$. Then G has derived length $d(G) \leq 5$.

Proof

Proceeding as in (4.4.1), G is soluble.

Let $X = \langle x \rangle$. Then $x^p \in Z(AX)$ since $X \lhd^2 AX$ implies that for all $a \in A$

$$\begin{aligned}[a, x^p] &= [a, x^{p-1}][a, x]^{x^{p-1}} \\ &= [a, x^{p-1}][a, x][a, x, x^{p-1}] \\ &= [a, x^{p-1}][a, x][a, x, x]^{p-1}, \text{ since } [A, X, X, X]=1 \\ &= [a, x]^p[a, x, x]^{(p-1)+(p-2)+\dots+1}.\end{aligned}$$

Thus $[a, x^p] = 1$, for all $a \in A$ (when $p \neq 2$) since A is elementary.

Suppose that $G''' \not\leq A\langle x^p\rangle B$, then $G''' \ni ax^{\alpha}b$ for some $a \in A$, $b \in B$ and $\alpha \not\equiv 0 \bmod p$. Thus

$$(x^{\alpha}b)^k \in AG'''$$

for all $k \in \mathbb{Z}$. Since X normalises B we see that this implies that

$$AG'''B \ni x^{\alpha k}$$

for all $k \in \mathbb{Z}$. Hence

$$AG'''B \supseteq AXB = G.$$

So $G''' \not\subseteq A\langle x^p\rangle B$ implies that $G = AG'''B$.

Then G/G''' is metabelian by Itô (see Notation) and so $G'' \leq G'''$ which is a contradiction because $G''' \neq 1$ and G is soluble.

Therefore we must have $G''' \subseteq A\langle x^p\rangle B$.

It follows that

$$G'''B = (A\langle x^p\rangle \cap G'''B)B$$

which is a product of two abelian subgroups since $x^p \in Z(AX)$.

Thus G''' is metabelian and $d(G) \leq 5$ as required.

$\square$

Corollary (4.4.3)

Let $G = AXB$ be a group with abelian 2-subgroups A, X and B where A is elementary, B has finite exponent and $X \cong C_{2^n}$ for $n \in \{1, 2\}$. Suppose that $A^X = A$, $B^X = B$, $X \lhd^2 AX$ and $X \lhd^2 XB$, then G has derived length $d(G) \leq 5$.

Proof

As usual, G is soluble by Corollary (4.4.1). If $n = 1$ then (4.4.1) gives the result $d(G) \leq 5$ immediately.

Suppose that $n = 2$ and write $\langle x\rangle = X \cong C_4$. Then

$$[a, x^2] = [a, x][a, x]^x = [a, x]^2[a, x, x]$$

and it follows that

$$[a, x, x] = [a, x^2]$$

for all $a \in A$ since A is elementary.

If $A \cap X = 1$ then $X \lhd^2 AX$ and so $[A, X, X] = 1$. Therefore $[a, x^2] = 1$ for all $a \in A$.

If $A \cap X \neq 1$ then $A \cap X = \langle x^2 \rangle$ since A is elementary.

Hence $x^2 \in Z(AX)$ in both cases. As in the proof of Corollary (4.4.2) $G''' \subseteq A\langle x^2 \rangle B$, otherwise

$$G = AG'''B$$

and so

$$G'' \leq G''' \neq 1$$

which is a contradiction.

Now, $G''' \subseteq A\langle x^2 \rangle B$ implies $G'''B = (G'''B \cap A\langle x^2 \rangle)B$, and thus G''' is metabelian by Ito (see Notation).

Hence the derived length of G is less than or equal to 5.

□

An immediate corollary of this result is

Corollary (4.4.4)

Let $G = AXB$ be a group with elementary abelian 2-subgroups A, X, and B where $|X| = 2$. Suppose that $A^X = A$ and $B^X = B$, then G has derived length ≤ 5.

Proof

Let $\langle x \rangle = X = C_2$. Then

$$1 = [a, x^2] = [a, x][a, x]^x$$
$$= [a, x]^2[a, x, x]$$

for all $a \in A$. Thus $[a, x, x] = 1$ for all $a \in A$ because A is elementary.

It follows that $[A, X, X] = 1 \leq X$ and so $X \triangleleft^2 AX$. Similarly $X \triangleleft^2 XB$.

Then $d(G) \leq 5$ by Corollary (4.4.3).

□

Corollary (4.4.5)

Let $G = AXB$ be a group with abelian 2-subgroups A, X and B where A and B are elementary and $X \cong C_{2^n}$ where $n \geq 3$. Suppose that $A^X = A$, $B^X = B$, $X \triangleleft^2 AX$ and $X \triangleleft^2 XB$. Then G has derived length $d(G) \leq 7$.

Proof

G satisfies the hypotheses of Corollary (4.2.3) with $d = 5$ (by (2.2.1)) and hence $d(G) \leq 7$.

□

Remark

Suppose $G = AXB$ is a group with abelian subgroups A, X and B, where X is finite and normalises both A and B. Then if B is finite we may apply the results of Chapters 2 and 3 directly to the finite factor group $G/\mathrm{Core}_G A$ in order to obtain a bound for the derived length of G.

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TITLE

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